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FUNCTION THEORY IN SPACES OF UNIFORMLY CONVERGENT FOURIER SERIES

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ABSTRACT. We study spaces of continuous functions on the unit circle with uniformly convergent Fourier series and show they possess such Banach space properties as the Pełczyński property, the Dunford-Pettis property and the weak sequential completeness of the dual space. We also prove extensions of theorems of Mooney and Sarason from the Hardy space H^∞ to the space H^∞_U of bounded analytic functions whose partial Fourier sums are uniformly bounded.

1. INTRODUCTION

It is now well-known that many algebras of analytic functions, such as the ball-algebras and polydisk algebras, possess Banach space properties similar to those of $C(L)$ spaces. Furthermore, it has been found that knowledge of the Banach space structure of a space can often be applied to the function theory. In this paper we examine some spaces of continuous functions which fail to be algebras, yet we may apply many theorems to these spaces where the natural setting for our techniques was once thought to be algebras. The result is that these spaces are shown to possess certain Banach space properties shared by all $C(L)$ spaces. Furthermore, the Banach space theory can be applied to the function theory and we find that some well-known results concerning algebras of functions hold for these spaces as well.

Let Γ be the unit circle in the complex plane and let m be normalized Lebesgue measure on Γ. For $f \in L^1(m)$ we have $\hat{f}(n) = \int f\bar{z}^n\,dm$. We define the Banach space U_C to be the space of continuous functions f with the property that the partial sums $P_n f = \sum_{k=-n}^{n} \hat{f}(k)z^k$ converge to f uniformly on Γ. If $f \in U_C$, then $\|f\|_{U_C} = \sup_{n\geq 0} \|P_n f\|_\infty$ is a complete norm on U_C. We define the closed subspace U_A to be those functions $f \in U_C$ such that $\hat{f}(n) = 0$ for $n < 0$.

J. Bourgain discovered a great deal about the Banach space structure of U_A in [1] where he proved that U_A shares many properties with spaces of the form $C(L)$ for compact spaces L. However, U_A fails to be isomorphic to even a quotient of a $C(L)$ space (see for example [11]). While techniques used by Bourgain and others

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have been successfully applied to many algebras of functions, the spaces U_A and U_C fail to be subalgebras of $C(\Gamma)$ (see [7]). The key tools used by Bourgain in [1] are the very deep results of Carleson concerning the pointwise convergence of Fourier series in L^2. However, many of Bourgain's results are reproved in [11] using little harmonic analysis. In [11], the author studied the space U_A and did not consider U_C. The same is true of Bourgain in [1]. In this paper we find that the results obtained for U_A can be proved just as easily for U_C and as far as we know these results for U_C are new.

In this paper, we will be concerned more with function theory rather than Banach space theory. Nevertheless, the results concerning the Banach space structure of these spaces will be of paramount importance. Certain types of results from function theory can be derived from characterizations of weak compactness in the dual of the space. One property we consider is defined as follows. A *weak Cauchy* sequence in a Banach space X is a sequence $\{x_n\}$ with the property that $\lim x^*(x_n)$ exists for every x^* in the dual space X^*. The space X is *weakly sequentially complete* if every weak Cauchy sequence converges in the weak topology to some element of X. For example, every $L^1(\mu)$-space is weakly sequentially complete. A related theorem from function theory is Mooney's theorem from [8]. Let A be the disk algebra, the subalgebra of $C(\Gamma)$ of functions which extend to be analytic in the open disk. Let H^∞ be the Hardy space of boundary values of bounded analytic functions in the open unit disk which is also the weak-star closure of A in $L^\infty(m)$. Mooney's theorem states that if $\{f_n\}$ is a sequence in $L^1(m)$ such that $\lim \int f_n h$ exists for every $h \in H^\infty$, then there exists a function $f \in L^1$ such that $\lim \int f_n h = \int fh$ for every $h \in H^\infty$.

Mooney's theorem is related to weak sequential completeness as follows. The Hardy space H^1 is the closure in $L^1(m)$ of the disk algebra, and H^1_0 is the subspace of functions vanishing at the origin, i.e., of functions f with $\hat{f}(0) = 0$. It is easy to see that the predual of H^∞ is isometrically isomorphic to L^1/H^1_0 and that Mooney's theorem is equivalent to the weak sequential completeness of L^1/H^1_0. Furthermore, the F. and M. Riesz theorem states that the annihilator $A^\perp$ of A is equal to H^1_0 and therefore A^* is isometrically isomorphic to $L^1/H^1_0 \oplus_{l^1} L^1(\mu)$ where $L^1(\mu)$ is isomorphic to the space of measures singular to Lebesgue measure. Mooney's theorem is therefore equivalent to the weak sequential completeness of A^*.

A stronger property than the weak sequential completeness of the dual is defined as follows. Recall that a linear operator between Banach spaces $T : X \to Y$ is *weakly compact* if the weak closure of the image of the unit ball under T is weakly compact in Y. We say a Banach space X has the *Pełczyński property* or *property (V)* if when $T : X \to Y$ is a continuous linear operator that fails to be weakly compact, then T is an isomorphism when restricted to an isomorphic copy of c_0 in X. It is well-known that if X has the Pełczyński property then X^* is weakly sequentially complete. Bourgain proved U_A has the Pełczyński property in [1] using the hard harmonic analysis tools mentioned above, while a simpler proof was given in [11]. We show in this paper that U_C has the Pełczyński property as well, the proof being very similar to the proof for the space U_A. In addition to the Pełczyński property we use a technique of Bourgain (see Theorem 2.1) to give characterization of weakly compact sets in the dual space which is of a more local nature. Essentially, a bounded set in the dual is weakly compact if and only if it fails to contain the unit vector basis of l^1_n uniformly.

Recall that a linear operator $T : X \to Y$ is *completely continuous* if when $x_n \xrightarrow{w} 0$ in X (that is, tends to zero in the weak topology), then $\|Tx_n\| \longrightarrow 0$. The space X has the *Dunford-Pettis property* if every weakly compact operator from X is completely continuous. It is well-known that X has the Dunford-Pettis property whenever X^* does. $L^1(\mu)$-spaces and $C(K)$ spaces are the basic examples of spaces with the Dunford-Pettis property. It was shown in [11] that U_A and its dual have the Dunford-Pettis property, and these results are extended to U_C in this paper. The details can be found in Section 2 where the results are summarized in Theorem 2.2.

The primary tool used to prove the results above is the Hankel-type operator. If $X \subset C(L)$ is a closed subspace and $g \in C(L)$, the corresponding Hankel-type operator S_g maps X to $C(L)/X$ by taking f to $gf + X$. All of the Banach space properties mentioned above are determined by knowing when the operators S_g have such traits as weak compactness or complete continuity. These Hankel-type operators were studied by B. Cole and T.W. Gamelin in [4] where the authors restricted their attention to uniform algebras (closed subalgebras of $C(L)$ containing the constants and separating the points of L) rather than arbitrary subspaces of $C(L)$. Cole and Gamelin wanted to know when the operators S_g were weakly compact and found that for algebras of analytic functions in several complex variables this was closely related to the solvability of a $\bar{\partial}$-problem with a mild gain smoothness. Furthermore, they showed that for an arbitrary uniform algebra A the weak compactness of the operators S_g is equivalent to a space of the form $H^\infty + C$ being a closed subalgebra of an L^∞-space. This generalization of Sarason's theorem, which we apply to U_A and U_C, is further discussed below.

Many connections have been found between the Banach space structure of X and the properties of the operators S_g. It can be deduced from Bourgain's work in [3] that if S_g^{**} is completely continuous for every $g \in C(L)$, then X and X^* have the Dunford-Pettis property. In [11] the author proved that if S_g is weakly compact for every g, then X has the Pełczyński property and X^* is weakly sequentially complete. We provide a relatively easy proof that U_A and U_C are isomorphic to subspaces X and Y, respectively, of some $C(K)$ space such that the Hankel-type operators S_g are all weakly compact and have completely continuous second adjoints. This was already proven in [11] for U_A and in this paper we extend the results to U_C.

In Section 3 we study an extension of Mooney's theorem which was stated in Bourgain's paper without proof. In Mooney's theorem we replace H^∞ by the space H_U^∞ consisting of H^∞ functions whose Fourier partial sums are uniformly bounded, and furthermore insist that the sequence $\{f_n\}$ be bounded in the L^1-norm. We give a proof of this result and show that the boundedness assumption cannot be removed. A version of this theorem for the space L_U^∞ of L^∞ functions with uniformly bounded Fourier sums is also presented. There are two main ingredients for the proof of this result, one being the weak sequential completeness of the dual of U_A which requires virtually no harmonic analysis and the other being a very deep theorem due to D. Oberlin. The key ingredient in the proof of Oberlin's theorem is a result of Vinogradov which in turn depends to a significant extent on Carleson's theorem concerning the pointwise convergence of Fourier series in L^2.

In Section 4 we prove a result for the space H_U^∞ which, as in the case of Mooney's theorem, is very similiar to a well-known theorem for H^∞. The theorem we speak of is Sarason's theorem which states that $H^\infty + C(\Gamma)$ is a closed subalgebra of L^∞. Since H_U^∞ is not an algebra, we cannot hope to extend Sarason's theorem directly to H^∞_U, but we show the next best thing. If $h \in H^\infty_U$ and $g \in C(\Gamma)$, we prove that $gh \in H^\infty_U + C(\Gamma)$. In addition to this, we show that a similar phenomenon occurs when multiplying the partial sums of the Fourier series of h simultaneously by a convergent sequence of continuous functions. The result is contained in Theorem 4.1.

In the sequel, Γ will denote the unit circle and m will be normalized Lebesgue measure on Γ. If $g \in L^1(m)$, we define $P_n g = \sum_{k=-n}^{n} \hat{g}(k) z^k$. If L is compact topological space, then $C(L)$ is the Banach space of continuous complex valued functions on L with the supremum norm and $M(L) = C(L)^*$. If X is a Banach space, then B_X denotes the closed unit ball of X.

2. Banach space properties of U_A and U_C

The results in this section concerning the linear topological structure of the Banach space U_A first appeared in [1]. The proof in [1] required a significant amount of hard harmonic analysis such as Carleson's theorem on the pointwise convergence of Fourier series. More precisely, it is Carleson's methods that are being used by Bourgain. Many of the results which appeared in [1] were reproved in [11] using essentially no harmonic analysis.

In this section we will review the results from [11] and show that they can be applied to the space U_C as well. However, we will use a slightly different technique which yields results of a more "local" nature.

Suppose L is a compact space and $X \subset C(L)$ is a closed subspace. If $g \in C(L)$, we define $S_g : X \to C(L)/X$ by $f \mapsto gf + X$. It was proved in [11] that if S_g is weakly compact for every $g \in C(L)$, then X has the Pełczyński property. In [3], it was proved that if S_g^{**} is completely continuous for every g, then X and X^* have the Dunford-Pettis property.

Recall that if X and Y are Banach spaces, then a linear operator $T : X \to Y$ is said to be *absolutely summing* or *1-summing* if there exists a constant C with

$$\sum \|Tx_n\| \le C \sup_{x^* \in B_{X^*}} \sum |x^*(x_n)|$$

for all finite sequences $\{x_n\}$. The prototypical 1-summing operator is the natural inclusion $X \hookrightarrow L^1(m)$ where X is a closed subspace of $C(L)$ for some compact space L and m is some element of $M(L)$. It is well-known that 1-summing operators are weakly compact and completely continuous and have second adjoints which are 1-summing. For more information on 1-summing operators, see [12].

In [2], Bourgain's work may be interpreted as the study of operators of the form S_g which behave like 1-summing operators. Let m be a probability measure on a compact space L and let $X \subset C(L)$ be a closed subspace. Suppose $T : X \to Y$ is a linear operator where Y is any Banach space. We shall say T is *nearly dominated by m* (or just *nearly dominated* when m exists) if whenever $\{f_n\} \subset X$ is a uniformly bounded sequence such that $\int |f_n|\, dm \longrightarrow 0$, then $\|Tf_n\| \longrightarrow 0$. If we were to remove the uniform boundedness of $\{f_n\}$ from the definition, then this property would imply T is a 1-summing operator. It follows from the Pietsch factorization theorem (cf. [12], III.F.8) that 1-summing operators are nearly dominated and we leave it as an exercise to the reader to show nearly dominated operators are weakly compact and have completely continuous second adjoints (cf. Lemma 2.3 in [11]).

It follows from the results mentioned in the discussion above that if the operator S_g is nearly dominated for every g (where the measure m may depend on g), then X

has the Pełczyński property, X^* is weakly sequentially complete and X and X^* have the Dunford-Pettis property. Furthermore, the following theorem can be deduced from the work of Bourgain in [2] (see also [12], III.D.31). Although Bourgain studies spaces where the S_g operators are nearly dominated by a fixed measure, it is easy to see from the proof that the measure may vary with g. However, the two primary examples considered in this paper will each possess a single measure that nearly dominates each S_g. In the language of Wojtaszczyk in [12], X is then a *rich subspace* of $C(K)$.

Theorem 2.1. *Let L be a compact space and let $X \subset C(L)$ be a closed subspace. Suppose the operator S_g is nearly dominated for every $g \in C(L)$. Then the weak closure of a bounded subset $E \subset X^*$ fails to be weakly compact if and only if there exists a constant $C > 0$ such that E contains a C-copy of the unit vector basis of l^1_n for $n = 1, 2, 3, \ldots$.*

The results concerning the Pełczyński property and the Dunford-Pettis property for the space U_A were discussed in [11]; however Bourgain's theorem on uniform copies of l^1_n was not. It follows from the results in [11] that U_A is isometrically isomorphic to a subspace $X \subset C(K)$ such that the operator S_g is nearly dominated by a fixed measure m for each $g \in C(K)$. The key trick is the observation from [10] that if m is a probability measure on K, then the set of $g \in C(K)$ with the property that S_g is nearly dominated by m form a closed subalgebra of $C(K)$. The same results hold with nearly identical proofs for the space U_C and for the sake of completeness we give a proof for U_C.

Let $K' = \{1/n\}_{n=1}^{\infty} \cup \{0\}$ and let $K = K' \times \Gamma$. Define a sequence of closed subspaces of K by $\Gamma_n = \{1/(n+1)\} \times \Gamma$ for $n \geq 0$ and let $\Gamma_\infty = \{0\} \times \Gamma$. If $\Phi \in C(K)$, we can write $\Phi = (\varphi_\infty, \varphi_0, \varphi_1, \varphi_2, \ldots)$ where $\Phi|_{\Gamma_n} = \varphi_n$ for $n \geq 0$, $\Phi|_{\Gamma_\infty} = \varphi_\infty$ and $\varphi_n \longrightarrow \varphi_\infty$ uniformly. Define an isometry $i : U_C \to C(K)$ by $i(F) = (F, P_0(F), P_1(F), P_2(F), \ldots)$ and let $Y = i(U_C)$. We also denote by m the measure on K which is normalized Lebesgue measure on Γ_∞ so that $F \in U_C$ implies $\hat{F}(n) = \int_K i(F)\bar{z}^n \, dm$. We define $\widehat{i(F)}(n) = \hat{F}(n)$ and $P_n(i(F)) = P_n(F)$.

Claim. The operator $S_g : Y \to C(K)/Y$ is nearly dominated by m for every $g \in C(K)$. Note that the functions $(0, z, 0, 0, \ldots)$, $(0, 0, z, 0, 0, \ldots), \ldots, (z, z, z, \ldots)$ and their conjugates separate the points of K. Therefore, by the Stone-Weierstrass theorem and the observation above, it suffices to show the operator S_g is nearly dominated for each of these functions. One easily verifies that if $f \in Y$ and g is one of the above functions, then gf differs from an element of Y by a function $\Phi = (\varphi_\infty, \varphi_0, \varphi_1, \ldots) \in C(K)$ with the property that each φ_k has the form $\sum_{k=-n}^{n} a_k \hat{f}(k) z^k$ where $\sup |a_k| \leq M$ and M and n depend only on g. Hence $\|S_g f\| \leq nM \, \|f\|_{L^1(m)}$ and so S_g is actually 1-summing for these particular g.

In summary, we have the following.

Theorem 2.2. *Let Z be the space U_A or U_C. Then the following hold.*

(a) Z and Z^* have the Dunford-Pettis property.

(b) Z has the Pełczyński property.

(c) Z^* is weakly sequentially complete.

(d) If $E \subset Z^*$ is a bounded subset, then the following are equivalent: (i) the weak closure of E fails to be weakly compact; (ii) there exists a constant $C > 0$ such that E contains a C-copy of the unit vector basis of l^1_n for $n = 1, 2, 3, \ldots$.

For the space U_A, the results (b) and (c) first appeared in [1] and (a)–(c) were proved in [11] using the above techniques. It follows from (d) and a result of Pisier (see [12], III.C.16) that the reflexive subspaces of U_A^* and U_C^* have finite cotype. A stronger version of this for the space U_A appears in [1] where it is shown that the reflexive subspaces of U_A^* embed in L^p for some $1 \leq p < 2$ and therefore have cotype 2. As far as we know, the results for U_C are new.

3. An extension of Mooney's theorem to H_U^∞

The following theorem was offered in [1] without proof. Let m be normalized Lebesgue measure on the unit circle and let H^∞ be the Hardy space of boundary values of bounded analytic functions in the unit disk. We define a Banach space

$$H_U^\infty = \left\{ h \in H^\infty \;\middle|\; \sup_{n \geq 0} \|P_n h\|_\infty < \infty \right\}$$

with $\|h\|_{H_U^\infty} = \sup_{n\geq 0} \|P_n h\|_\infty$.

Theorem 3.1. *Suppose $\{f_n\} \subset L^1(m)$ is a sequence with $\sup \|f_n\|_1 < \infty$ such that $\lim \int f_n h \, dm$ exists for each $h \in H_U^\infty$. Then there exists an $f \in L^1(m)$ such that*

$$\lim \int f_n h \, dm = \int f h \, dm$$

for every $h \in H_U^\infty$.

If we remove the boundedness assumption on $\{f_n\}$ and replace H_U^∞ with H^∞, then we have the theorem of Mooney from [8]. Mooney's theorem is equivalent to the weak sequential completeness of the dual of the disk algebra, and one of the principal ingredients in Theorem 3.1 is the weak sequential completeness of the dual of U_A. Although our proof of the weak sequential completeness of U_A^* requires virtually no harmonic analysis, there is a critical ingredient for the proof of Theorem 3.1 which is a very difficult theorem from harmonic analysis and that is the following theorem of Oberlin from [9]. Let K be the space from Section 2 and let $i : U_A \to C(K)$ be the corresponding isometric embedding. Let $X = i(U_A)$.

Theorem 3.2. *Suppose $\mu \in X^\perp$. Then $\mu|_{\Gamma_\infty}$ is absolutely continuous with respect to Lebesgue measure on Γ_∞.*

Using the techniques from Chapter 2, Section 12 of [5], Oberlin obtains the following extension of the Rudin-Carleson theorem. If E is a closed subset of Lebesgue measure zero on the unit circle and if g is a continuous function on E, then there exists an $f \in U_A$ such that $f|_E = g$ and $|f(z)| < \sup \{ |g(w)| \mid w \in E \}$ if $z \notin E$. In fact, a somewhat stronger result holds where we may interpolate g and dominate the partial sums of the Fourier series of f as well. We leave it to the reader to work out the details.

We will supply a proof of Bourgain's theorem on H_U^∞ as well as a version for the space L_U^∞ of L^∞ functions with uniformly bounded Fourier series. Let $E = \bigcup \Gamma_n \subset K$ and also denote by m normalized Lebesgue measure on Γ_∞. Define $\mathcal{B}_0 = M(E) \oplus_{l^1} L^1(m) \subset M(K)$. That is, $\mathcal{B}_0$ is the space of measures on K whose restriction to Γ_∞ is absolutely continuous with respect to m. If $\mu \in M(K)$, we may write $\mu = \mu^{(\infty)} + \sum \mu^{(n)}$ where $\mu^{(\infty)} = \mu|_{\Gamma_\infty}$ and $\mu^{(n)} = \mu|_{\Gamma_n}$. Let $\mu^{(\infty)} = f_\mu \, dm + \mu_s$ be the Lebesgue decomposition of μ on Γ_∞.

Proposition 3.3. *If $\mu \in \mathcal{B}_0$ and $h \in H_U^\infty$, then the action*

$$\left\langle h, \mu + \mathcal{B}_0 \cap X^\perp \right\rangle = \int h f_\mu \, dm + \sum \int (P_n h) \, d\mu^{(n)}$$

defines an isometric isomorphism of H_U^∞ onto $(\mathcal{B}_0/\mathcal{B}_0 \cap X^\perp)^*$.

Proof. We first show the action is well-defined. Let

$$\langle h, \nu \rangle = \int h f_\nu \, dm + \sum \int (P_n h) \, d\nu^{(n)}$$

for $h \in H_U^\infty$ and $\nu \in \mathcal{B}_0$.

Claim. $\lim_{n\to\infty} \langle P_n h, \nu \rangle = \langle h, \nu \rangle$. This follows from the inequality

$$\left| \sum_{k=n}^{\infty} \int P_k h \, d\nu^{(k)} \right| \le \|h\|_{H_U^\infty} \sum_{k=n}^{\infty} \left\| \nu^{(k)} \right\|$$

and the fact that $P_n h \xrightarrow{w^*} h$ in $L^\infty(m)$ on the unit circle. Therefore, since $\langle P_n h, \nu \rangle = \int_K i(P_n h) \, d\nu$, it follows that $\langle h, \nu \rangle = 0$ when $\nu \in \mathcal{B}_0 \cap X^\perp$.

We now have a linear map $T : H_U^\infty \to (\mathcal{B}_0/\mathcal{B}_0 \cap X^\perp)^*$ defined by

$$\left\langle Th, \mu + \mathcal{B}_0 \cap X^\perp \right\rangle = \left\langle h, \mu + \mathcal{B}_0 \cap X^\perp \right\rangle.$$

Claim. T is an isometry. If $\mu \in \mathcal{B}_0$ and $\left\| \mu + \mathcal{B}_0 \cap X^\perp \right\| \le 1$, then we may assume $\|\mu\| \le 1 + \varepsilon$ so

$$\left| \left\langle h, \mu + \mathcal{B}_0 \cap X^\perp \right\rangle \right| \le \|h\|_{H_U^\infty} (1 + \varepsilon)$$

for $h \in H_U^\infty$. Hence, $\|Th\| \le \|h\|_{H_U^\infty}$. Since

$$\|P_n h\|_\infty = \sup_{g \in L^1, \|g\|_1 \le 1} \left| \int (P_n h) g \, dm \right|$$

and since we may regard $g \, dm$ as a measure ν on $\Gamma_n \subset K$ so that $\|\nu\| = \|g\|_1$ and

$$\left\langle h, \nu + \mathcal{B}_0 \cap X^\perp \right\rangle = \int (P_n h) g \, dm,$$

it follows that $\|h\|_{H_U^\infty} = \|Th\|$ and T is an isometry.

We will now show that T is surjective. Let $x^* \in (\mathcal{B}_0/\mathcal{B}_0 \cap X^\perp)^*$ and let $f_k(z) = \bar{z}^k$ for $z \in \Gamma$. Let $c_k = \left\langle x^*, f_k \, dm + \mathcal{B}_0 \cap X^\perp \right\rangle$ and let $\alpha \in \Gamma$. Then

$$\left| \sum_{k=0}^{n} c_k \alpha^k \right| \le \|x^*\| \left\| \left(\sum_{k=0}^{n} \alpha^k f_k \, dm \right) + \mathcal{B}_0 \cap X^\perp \right\|. \tag{3.1}$$

Claim. For each $\alpha \in \Gamma$ and $n = 0, 1, 2, \ldots$ we have

$$\left\| \left(\sum_{k=0}^{n} \alpha^k f_k \, dm \right) + \mathcal{B}_0 \cap X^\perp \right\| \le 1.$$

By the translation invariance of the space U_A it suffices to let $\alpha = 1$. Given $n \ge 0$ we choose $f = -\sum_{k=0}^{n} \bar{z}^k$ and μ to be the point mass at 1 on Γ_n. If $g \in X$, then

$g|_{\Gamma_n} = P_n g$ so that $\langle P_n g, \mu\rangle = \sum_{k=0}^n \hat{g}(k)$. Hence $f\,dm + \mu \in X^\perp \cap \mathcal{B}_0$ and

$$\left\| \sum_{k=0}^n f_k\,dm + (f\,dm + \mu) \right\| = 1,$$

proving the claim.

If we define $h \in H^\infty_U$ by $\hat{h}(n) = c_n$, then it follows from (3.1) that $\|h\|_{H^\infty_U} \leq \|x^*\|$. We must now show that $Th = x^*$. Recall that the Hardy space H^1_0 is the annihilator of the disk algebra. The natural inclusion map $L^1(m) \hookrightarrow \mathcal{B}_0$ induces an injection $L^1/H^1_0 \overset{\sigma}{\hookrightarrow} \mathcal{B}_0/\mathcal{B}_0 \cap X^\perp$.

Claim. σ has dense range. It suffices to show that if $\mu \in M(\Gamma_n)$, then $\mu + \mathcal{B}_0 \cap X^\perp \in R(\sigma)$. If $f \in X$ we have

$$\int f\,d\mu = \int P_n f\,d\mu = \int fQ\,dm$$

where $Q = \sum_{k=0}^n \hat{\mu}(-k)\bar{z}^k$ so $\mu + \mathcal{B}_0 \cap X^\perp = \sigma(Q\,dm + H^1_0)$, proving the claim. Since the trigonometric polynomials are dense in $L^1(m)$, it follows that

$$\overline{\mathrm{sp}}\left\{\bar{z}^k\,dm + \mathcal{B}_0 \cap X^\perp\right\}_{k=0}^\infty = \frac{\mathcal{B}_0}{\mathcal{B}_0 \cap X^\perp}. \tag{3.2}$$

Furthermore, for $k = 0, 1, 2, \ldots$ we have

$$\left\langle \bar{z}^k\,dm + \mathcal{B} \cap X^\perp, Th\right\rangle = \hat{h}(k) = c_k = \left\langle \bar{z}^k\,dm + \mathcal{B} \cap X^\perp, x^*\right\rangle,$$

and so $x^* = Th$, finishing the proof of the proposition.

□

Recall that a *band of measures* in $M(K)$ is a closed subspace $\mathcal{B}$ of $M(K)$ with the property that when $\mu \in \mathcal{B}$ and $\nu << \mu$, then $\nu \in \mathcal{B}$. If $\mathcal{B}$ is a band, then every measure $\nu \in M(K)$ can be decomposed as $\nu = \nu_a + \nu_s$ where $\nu_a \in \mathcal{B}$ and ν_s is singular to every element of $\mathcal{B}$. Let $\mathcal{B}_{X^\perp}$ be the band generated by $X^\perp$ so that $M(K) = \mathcal{B}_{X^\perp} \oplus_{l^1} \mathcal{S}_{X^\perp}$ where $\mathcal{S}_{X^\perp}$ is the band of measures singular to $\mathcal{B}_{X^\perp}$. It follows that X^* is isometrically isomorphic to $\mathcal{B}_{X^\perp}/X^\perp \oplus_{l^1} \mathcal{S}_{X^\perp}$.

Using the proof of the fact that the map σ defined above has dense range we see that $M(E) \subset \mathcal{B}_{X^\perp}$. Since $z\,dm \in X^\perp$, we have $\mathcal{B}_0 \subset \mathcal{B}_{X^\perp}$. We must now use Oberlin's theorem. Oberlin's theorem implies the converse of our inclusion so that we have $\mathcal{B}_{X^\perp} = \mathcal{B}_0$. We may now prove the main theorem.

Proof of Theorem 3.1. Let $\{f_n\}$ be as in the theorem. Then, since $(\mathcal{B}_{X^\perp}/X^\perp)^* \cong H^\infty_U$, it is a consequence of the weak sequential completeness of X^* that $f_n\,dm + X^\perp \xrightarrow{w} \mu + X^\perp$ (convergence in the weak topology) in $\mathcal{B}_{X^\perp}/X^\perp$ for some $\mu \in \mathcal{B}_{X^\perp}$. Furthermore, the boundedness of $\{f_n\}$ and the limit condition imply $f_n + H^1_0 \xrightarrow{w^*} \nu + H^1_0$ in $M(\Gamma)/H^1_0$ for some $\nu \in M(\Gamma)$. Let $\nu|_{\Gamma_\infty}$ denote the measure ν regarded as an element of $M(K)$ supported on Γ_∞. Using (3.2) we have

$$\overline{\mathrm{sp}}\left\{\bar{z}^k\,dm + X^\perp\right\} = \mathcal{B}_{X^\perp}/X^\perp$$

so that $\nu|_{\Gamma_\infty} + X^\perp = \mu + X^\perp$. Hence $\nu << m$ by Oberlin's theorem. Let $f = d\nu/dm$. □

Unlike Mooney's original theorem, the assumption that $\{f_n\}$ be bounded in L^1 cannot be removed from Theorem 3.1.

Proposition 3.4. *The injection $L^1/H^1_0 \overset{\sigma}{\hookrightarrow} \mathcal{B}_{X^\perp}/X^\perp$ fails to be surjective. Hence, the condition that $\{f_n\}$ be bounded cannot be removed from Theorem 3.1.*

Proof. If σ were surjective, there would exist a constant C such that

$$\max_{h \in H^\infty_U, \|h\|_{H^\infty_U} \le 1} \left| \int f h \, dm \right| \ge C \sup_{g \in A, \|g\|_\infty \le 1} \left| \int f g \, dm \right| \tag{3.3}$$

for all $f \in L^1(m)$ where A is the disk algebra. Let $\alpha \in \Gamma$ and let $n \ge 0$ be an integer. Let $f = \sum_{k=0}^n \alpha^k \bar{z}^k$. Then (3.3) implies $\sup_{g \in A, \|g\|_\infty \le 1} |P_n g(\alpha)| \le C^{-1}$ so that $\sup_n \|P_n\|_A \le C^{-1}$ where $\|P_n\|_A$ is the norm of P_n as an operator on the disk algebra. We leave it to the reader to verify this is a contradiction.

We found above that σ has dense range. Therefore, if we let $\mu + X^\perp$ be an element in $\mathcal{B}_{X^\perp}/X^\perp \backslash R(\sigma)$, then there exists a sequence $\{f_n\} \subset L^1$ with $f_n \, dm + X^\perp \longrightarrow \mu + X^\perp$. The sequence $\{f_n\}$ now satisfies the limit assumption while the conclusion of the theorem fails. □

All of the results above have versions for the space U_C. Let L^∞_U be the subspace of $L^\infty(m)$ consisting of functions f such that $\sup \|P_n f\|_\infty < \infty$ with $\|f\|_{L^\infty_U} = \sup \|P_n f\|_\infty$. Recall the isometry $i : U_C \to C(K)$ where $Y = i(U_C)$. Then $X \subset Y$ and so Oberlin's theorem applies to Y. In fact, we find that $\mathcal{B}_{Y^\perp} = \mathcal{B}_{X^\perp}$ and $(\mathcal{B}_{Y^\perp}/Y^\perp)^* \cong L^\infty_U$ with an isometric isomorphism, the action being similar to that of H^∞_U on $\mathcal{B}_{X^\perp}/X^\perp$. The next result is proved in the same manner as the above.

Proposition 3.5. *Suppose $\{f_n\} \subset L^1$ with $\sup \|f_n\|_1 < \infty$ and $\lim \int f_n g \, dm$ exists for every $g \in L^\infty_U$. Then there exists an $f \in L^1(m)$ so that $\lim \int f_n g \, dm = \int f g \, dm$ for every $g \in L^\infty_U$. Furthermore, we cannot remove the boundedness of $\{f_n\}$ from the hypothesis.*

The weak sequential completeness of the dual can be used to make a remark about interpolation by H^∞_U functions. We say a sequence $\{z_n\}$ in the open unit disk is interpolating for H^∞_U if whenever $\{\alpha_n\} \in l^\infty$ there exists an $h \in H^\infty_U$ such that $h(z_n) = \alpha_n$. In general, a bounded sequence $\{x_n\}$ in a Banach space X is a basic sequence equivalent to the unit vector basis of l^1 if and only if the sequence interpolates the dual space X^*, that is, if and only if the map $T : X^* \to l^\infty$ by $(Tx^*)(n) = x^*(x_n)$ is surjective. By the Rosenthal-Dor theorem, every bounded subset in a weakly sequentially complete space which fails to have a weak closure which is weakly compact contains a subsequence equivalent to the unit vector basis of l^1. If $\{z_n\}$ is a sequence in the open unit disk tending toward the boundary, then the corresponding sequence of point evaluations on U_A are easily seen to be a bounded subset of $\mathcal{B}_{X^\perp}/X^\perp \subset (U_A)^*$ which fails to have a weakly compact weak closure. We have now proved the following.

Proposition 3.6. *If $\{z_n\}$ is a sequence in open unit disk tending toward the boundary, then $\{z_n\}$ has a subsequence which is interpolating for H^∞_U.*

More information on interpolation by H^∞_U functions can be found in [6].

4. The space $H^\infty_U + C(K)$

Sarason's theorem on the unit circle states that $H^\infty + C(\Gamma)$ is a closed subalgebra of L^∞. B. Cole and T.W. Gamelin investigated extensions of this property to the setting of arbitrary uniform algebras in [4]. Some of their results actually hold for subspaces of $C(L)$ as opposed to subalgebras. These ideas, as well as Oberlin's theorem, will be used in the proof of the following theorem which is the main result of this section.

Theorem 4.1. *Suppose $h \in H_U^\infty$ and $\{g_n\}$ is a convergent sequence in $C(\Gamma)$ with limit g. Then there exist an $H \in H_U^\infty$ and functions $\{G_n\} \subset C(\Gamma)$ with $G_k \longrightarrow G$ in $C(\Gamma)$ so that $gh = H + G$ and $g_k(P_k h) = P_k H + G_k$ for $k = 0, 1, 2, \dots$.*

A representation of the second dual of U_A will be helpful. If $\mathcal{B}$ is a band of measures, then $\mathcal{B}^*$ is isometrically isomorphic to the space

$$L^\infty(\mathcal{B}) = \{(F_\mu)_{\mu\in\mathcal{B}} \,|\, F_\mu \in L^\infty(\mu), F_\nu = F_\mu \text{ a.e. } [\nu] \text{ when } \nu << \mu\}.$$

We denote an element of $L^\infty(\mathcal{B})$ by F, the norm being defined as

$$\|F\| = \sup_{\nu\in\mathcal{B}} \|F_\nu\|_\infty .$$

The action is defined by $\langle \mu, F\rangle = \int F_\mu \, d\mu$ for $\mu \in \mathcal{B}$. If $X \subset C(L)$ is a closed subspace and $\mathcal{B} \subset M(L)$ is a band, we define $H^\infty(\mathcal{B})$ to be the weak-star closure of X in $L^\infty(\mathcal{B})$. It follows that $(\mathcal{B}_{X^\perp}/X^\perp)^* \cong H^\infty(\mathcal{B})$ and $X^{**} \cong H^\infty(\mathcal{B}_{X^\perp}) \oplus_{l^\infty} L^\infty(\mathcal{S}_{X^\perp})$ with isometric isomorphisms. For more information on bands of measures, see [4].

If $X \subset C(K)$ is the space from Section 3, then we have an isometric isomorphism $U : H^\infty(\mathcal{B}_{X^\perp}) \to H_U^\infty$. If $G \in H^\infty(\mathcal{B}_{X^\perp})$ and $UG = h$, then $G_m = h$ a.e. $[m]$ and $G_\nu = P_n h$ a.e. $[\nu]$ for every $\nu \in M(\Gamma_n)$.

Proof of Theorem 4.1. We may define an element of $C(K)$ by

$$\Phi = (g, g_0, g_1, g_2, \dots).$$

Since $S_\Phi : X \to C(K)/X$ is weakly compact (see Section 2) we have $S_\Phi^{**}(X^{**}) \subset C(K)/X$. Now $(C(K)/X)^* \cong X^\perp \subset \mathcal{B}_{X^\perp}$ and

$$(C(K)/X)^{**} \cong L^\infty(\mathcal{B}_{X^\perp})/H^\infty(\mathcal{B}_{X^\perp}).$$

Using Oberlin's theorem we may identify $H^\infty(\mathcal{B}_{X^\perp})$ with H_U^∞ and therefore identify h with an element of X^{**}. It follows that $\Phi h + H^\infty(\mathcal{B}_{X^\perp}) \in C(K)/X$ and so $\Phi h \in H^\infty(\mathcal{B}_{X^\perp}) + C(K)$. Using the identification a second time we have $\Phi h \in H_U^\infty + C(K)$ which proves the theorem. □

Concerning the fact that $H^\infty + C(\Gamma)$ is a closed subspace of L^∞, the corresponding result for H_U^∞ is that $H_U^\infty + C(K)$ is a closed subspace of $L^\infty(\mathcal{B}_{X^\perp})$. This follows from the fact that $H^\infty(\mathcal{B}_{X^\perp}) + C(K) = q^{-1}(C(K)/X)$ where $q : L^\infty(\mathcal{B}_{X^\perp}) \to L^\infty(\mathcal{B}_{X^\perp})/H^\infty(\mathcal{B}_{X^\perp})$ is the natural quotient map. Similar results hold for the space L_U^∞.

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