ON INTEGERS OF THE FORM $2^k \pm p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$

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Abstract. In this paper we prove that the set of positive odd integers which have no representation of the form $2^m \pm p^{\alpha}q^\beta$, where $p, q$ are distinct odd primes and $n, \alpha, \beta$ are nonnegative integers, has positive lower asymptotic density in the set of all positive odd integers.

1. Introduction

The problem of expressing an odd integer $m > 1$ in the form $2^m + p$, where $p$ is a prime and $n$ is a nonnegative integer, is an old one. Romanoff [8] showed that the set of positive odd numbers which can be expressed in the form $2^m + p$ has positive asymptotic density in the set of all positive odd numbers. P. Erdős [6] exhibited a residue class of odd integers each of which has no representation of the form $2^m + p$. Cohen and Selfridge [5] proved that there exist (infinitely many) odd numbers which are neither the sum nor the difference of a power of two and a prime power.

For a positive integer $n$ and an integer $a$, let

$$a(n) = \{a + nk : k \in \mathbb{Z}\}.$$  

We call $\{a_i(n_i)\}_{i=1}^t$ a covering system if every integer $y$ satisfies $y \equiv a_i \pmod{n_i}$ for at least one value of $i$. For the construction of covering systems one may refer to S. L. G. Choi [4]. For further related information one may see Guy [7], A19, B21 and F13.

For convenience we give the following definitions.

Definition 1. A positive integer $d$ is called a $(a, b)$-primitive divisor of order $n$ if $d | a^n - b^n$ and $d \nmid a^m - b^m$ for all $1 \leq m < n$.

Definition 2. $\{a_i(n_i)\}_{i=1}^t$ is called an $m$-covering system if every integer belongs to at least $m$ of $a_1(n_1), a_2(n_2), \cdots, a_t(n_t)$.

Definition 3. $\{a_i(n_i)\}_{i=1}^t$ is called a $(2, 1)$-primitive $m$-covering system if $\{a_i(n_i)\}_{i=1}^t$ is an $m$-covering system and there exist distinct primes $p_1, p_2, \cdots, p_t$ such that, for each $i$, $p_i$ is a $(2, 1)$-primitive divisor of order $n_i$ ($1 \leq i \leq t$).

For example $0(2), 3(4), 5(8), 9(16), 17(32), 33(64), 1(64)$ is a $(2, 1)$-primitive 1-covering system (corresponding primes are 3, 5, 17, 257, 65537, 641, 6700417 respectively).
From the form \( \pm 2^n \pm p^\alpha \) to \( \pm 2^n \pm p^\alpha q^\beta \) there are some essential difficulties. The first is to construct a \((2, 1)\)-primitive 2-covering system. We employ Choi’s method to complete such a construction. From this we need only consider \( p, q \) in a finite set for our purpose. The second difficulty is to give additional conditions as in the proof of Theorem 2 of [2] or to do a similar thing. To avoid this we employ a result of Baker to show that for \( p, q \) in a finite set, the set of integers of the form \( \pm 2^n \pm p^\alpha q^\beta \) is very thin in the set of all positive odd integers. For later use we give the argument in a general form.

In this paper the following results are proved.

**Theorem.** Suppose that there exists a \((2, 1)\)-primitive \( r \)-covering system. Then the set of odd positive integers which have neither the form \( 2^n + q_1^{\alpha_1} \cdots q_r^{\alpha_r} \) nor the form \( 2^n - q_1^{\alpha_1} \cdots q_r^{\alpha_r} \), where \( q_1, q_2, \cdots, q_r \) are distinct positive odd primes and \( n, \alpha_1, \cdots, \alpha_r \) are nonnegative integers, has positive lower asymptotic density.

**Corollary.** The set of odd positive integers which have neither the form \( 2^n + p^\alpha q^\beta \) nor the form \( 2^n - p^\alpha q^\beta \), where \( p, q \) are distinct positive odd primes and \( n, \alpha, \beta \) are nonnegative integers, has positive lower asymptotic density.

**Remark 1.** For given integers \( a, b \) with \( ab \) prime to corresponding primes \( p_1, \cdots, p_r \) in Definition 3, the same conclusion is true when one replaces odd integers by the integers prime to corresponding primes

\[ q \in \mathbb{Z} \backslash \{ 1 \} \] is a constant depending only on \( p, \alpha \).

Let \( x \rightarrow 0 \), we have

\[ |2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| \leq x \]

has at most \((2 \log x)^{r+1}\) nonnegative integral solutions \( n, \alpha_1, \cdots, \alpha_r \) for \( x \geq c \), where \( c \) is a constant depending only on \( p_1, \cdots, p_r \).

**Proof.** Suppose that \( n, \alpha_1, \cdots, \alpha_r \) are not all zero. Let

\[ \delta = (c_1(r+1))^{r+1} \prod_{i=1}^{r} \log p_i \cdot \log 2 \cdot \log \prod_{i=1}^{r} \log p_i, \]

for \( x \geq \delta \).
By Lemma 1 we have
\[
|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| = 2^n|1 - 2^{-n}p_1^{\alpha_1} \cdots p_r^{\alpha_r}|
\geq \frac{2^n}{(\max\{n, \alpha_1, \ldots, \alpha_r\})^\beta}
\]
and
\[
|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}|2^n - p_1^{-\alpha_1} \cdots p_r^{-\alpha_r} - 1|
\geq \frac{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}{(\max\{n, \alpha_1, \ldots, \alpha_r\})^\beta}.
\]
Thus
\[
|2^n - p_1^{\alpha_1} \cdots p_r^{\alpha_r}| = 2^n|1 - 2^{-n}p_1^{\alpha_1} \cdots p_r^{\alpha_r}|
\geq \frac{2^n}{(\max\{n, \alpha_1, \ldots, \alpha_r\})^\beta}.
\]
Then Lemma 2 follows immediately.

**Proof of the Theorem.** Suppose that \(\{a_i(n_i)\}_{i=1}^t\) is a \((2, 1)\)-primitive \(r\)-covering system and \(p_1, \ldots, p_t\) are corresponding primes in Definition 3. Take an integer \(M\) satisfying
\[
M \equiv 2^{a_i} \pmod{p_i}, \quad i = 1, \ldots, t.
\]
For any positive integer \(n\) there exist \(i_1, \ldots, i_r\) with \(1 \leq i_1 < i_2 < \cdots < i_r \leq t\) and
\[
n \in a_{i_j}(n_{i_j}), \quad j = 1, 2, \ldots, r.
\]
Then by (1) and
\[
2^{a_{i_j}} \equiv 1 \pmod{p_{i_j}}, \quad j = 1, 2, \ldots, r,
\]
we have
\[
M \equiv 2^n \pmod{p_{i_j}}, \quad j = 1, 2, \ldots, r.
\]
Thus
\[
M = 2^n + p_1^{\alpha_{i_1}} \cdots p_r^{\alpha_{i_r}} b, \quad \alpha_{i_j} > 0 \quad (j = 1, 2, \ldots, r), b \in \mathbb{Z}.
\]
Hence, if \(M\) has the form
\[
M = 2^n \pm q_1^{\beta_1} \cdots q_r^{\beta_r},
\]
where \(q_1, \ldots, q_r\) are primes, then \(q_i \in \{p_1, \ldots, p_t\} \quad (i = 1, 2, \ldots, r)\). It is clear that the number of integers \(M\) with
\[
M = 2^n + q_1^{\beta_1} \cdots q_r^{\beta_r} \leq x, \quad q_i \in \{p_1, \ldots, p_t\}, i = 1, 2, \ldots, r,
\]
is less than \(c'_t(2 \log x)^{r+1}\) for \(x \geq X_1\). By Lemma 2 the number of positive integers \(M\) with
\[
M = 2^n - q_1^{\beta_1} \cdots q_r^{\beta_r} \leq x, \quad q_i \in \{p_1, \ldots, p_t\}, i = 1, 2, \ldots, r,
\]
is less than \(c'_t(2 \log x)^{r+1}\) for \(x \geq X_2\). It is well known that the number of positive odd integers \(M\) with (1) and \(M \leq x\) is more than
\[
\frac{x}{2p_1 \cdots p_t} - 1, \quad x \geq X_3.
\]
Therefore, for 
\[ x \geq \max\{X_1, X_2, X_3\}, \]
there exist at least 
\[ \frac{x}{2p_1 \cdots p_r} - 1 - 2c_r(2 \log x)^{r+1} \]
positive odd numbers \( M \leq x \) which have neither the form \( 2^n + q_1^{\beta_1} \cdots q_r^{\beta_r} \) nor the form \( 2^n - q_1^{\beta_1} \cdots q_r^{\beta_r} \), where \( q_1, \ldots, q_r \) are primes. This completes the proof of the Theorem.

Proof of the Corollary. Since
\[ A = \{0(2), 3(4), 5(8), 9(16), 17(32), 33(64), 1(64)\} \]
and
\[ B = \{0(3), 4(9), 2(12), 10(18), 8(24), 16(36), 34(36), 20(48), 44(48), 1(5), 5(10), 7(20), 14(25), 13(30), 17(40), 9(50), 19(50), 23(60), 53(60), 37(80), 49(100), 77(120), 99(150), 157(160), 199(200), 199(300), 237(480), 157(240), 299(600)\} \]
are both 1-covering systems, \( A \cup B \) is a 2-covering system. By Birkhoff and Vandiver \[3\] (or Bang \[2\], Zsigmondy \[9\]), for \( n \geq 2, n \neq 6 \), there exist \((2, 1)\)-primitive prime divisors of order \( n \). Again, 641, 6700417 are \((2, 1)\)-primitive prime divisors of order 64; write this as 64 \( \leftrightarrow 641, 6700417 \). Similarly, 36 \( \leftrightarrow 37, 109; 48 \leftrightarrow 97, 673; 25 \leftrightarrow 601, 1801; 50 \leftrightarrow 251, 4051; 60 \leftrightarrow 61, 1321 \). Thus \( A \cup B \) is a \((2, 1)\)-primitive 2-covering system. Now the corollary follows from the theorem.

References