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EXTENSION THEOREMS FOR THE DISTRIBUTION SOLUTIONS TO $\mathcal{D}$-MODULES WITH REGULAR SINGULARITIES

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Dedicated to the Memory of Our Friend, E. Andronikof

ABSTRACT. Some extension theorems for the distribution solutions to $\mathcal{D}$-modules will be given. We will use the notion of regular singularities introduced by Kashiwara-Oshima (1977).

1. INTRODUCTION

In this paper, we will give several extension theorems for the distribution solutions (C^∞-solutions) to $\mathcal{D}$-modules. These results are the refinements of the results for hyperfunction solutions obtained in Kashiwara-Schapira [8] and [12] (cf. also [13]), for which we make use of the characterization in [7] of the systems with regular singularities (introduced also by Kashiwara-Oshima [7]) and the ideas of [4]. One of our results is considered as a variant of the main theorem of D'Agnolo-Tonin [4].

2. NOTATIONS AND KNOWN RESULTS

In the present paper, we essentially employ the notations and terminology of [9] and [1]. Let M be a real analytic manifold of dimension $n \geq 1$ and N a submanifold of codimension $d \geq 1$. We take a complexification $Y \subset X$ of $N \subset M$. We denote by $\mathcal{D}_X$ the sheaf of holomorphic differential operators on X, and consider the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions on M. Then we have the following results for hyperfunction solutions to coherent $\mathcal{D}_X$-modules.

Theorem 2.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

(i) (*Kashiwara-Schapira* [8]) *Assume $N = \{x_1 = 0\} \subset M$ is of codimention one in M and $\mathcal{M}$ is hyperbolic in the direction $+dx_1 \in \dot{T}^*_N M$. We set $\Omega = \{x_1 < 0\} \subset M$. Then we have an isomorphism:*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \xrightarrow{\sim} R\Gamma_\Omega R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N. \tag{2.1}$$

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(ii) ([12] *and* [13]) *We assume that the codimension d of N in M satisfies $d \geq 2$, and consider the irreducible decomposition* $\mathrm{char}\mathcal{M} = \bigcup_{j=1}^{m} V_j$ *of the characteristic variety of $\mathcal{M}$. Suppose that each V_j satisfies one of the following conditions:*

$$
\left\{ \begin{array}{l} \text{(a) } V_j \textit{ is non-characteristic for } Y \textit{ and elliptic, i.e. } \dot{T}^*_M X \cap V_j = \emptyset. \\ \text{(b) } V_j \textit{ is hyperbolic in the direction } \xi \in \dot{T}^*_N M. \end{array} \right. \tag{2.2}
$$

Then for every proper open convex cone $U \subset \dot{T}^*_N M$ containing ξ we have the isomorphism

$$
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \xrightarrow{\sim} \Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \tag{2.3}
$$

for an open tuboid $\Omega \subset M$ along N satisfying the condition $U^\circ \cap \dot{T}_N M \cap C_N(\Omega) = \emptyset$.

Let us give an example of Theorem 2.1 (ii), which will explain why it cannot be stated in the derived categories (that is, in all degrees). For details, see also the proof of Theorem 4.10 of [13].

Example 2.2 ([13])**.** We assume that $d = 2$ and let $Q \in \mathcal{D}_X$ be a hyperbolic differential operator in the direction $\xi \in \dot{T}^*_N M$. We take operators E_j $(j = 1, 2)$ such that the set $V_0 := \bigcap_{j=1,2} \{\sigma(E_j) = 0\} \subset T^*X$ is non-characteristic for Y and satisfies the ellipticity condition $\dot{T}^*_M X \cap V_0 = \emptyset$. Now we set $P_j = E_j Q +$(lower order terms). Then for the system $\mathcal{M} = \mathcal{D}_X / \sum_{j=1}^{2} \mathcal{D}_X P_j$ we have by Theorem 4.10 of [13]

$$
H^j[\mu_N R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)] \simeq 0 \quad \text{at} \quad \xi \quad \text{for} \quad j < d = 2, \tag{2.4}
$$

and a local Bochner type extension theorem for hyperfunction (real analytic) solutions as in Theorem 2.1 (ii) holds. Note that $\mathcal{D}$-modules $\mathcal{M}$ obtained in this way are not hyperbolic (in the direction $\xi \in \dot{T}^*_N M$), nor non-characteristic for Y in general. From the proof of Theorem 4.10 of [13], we cannot expect for all j the vanishing of cohomologies $H^j[\mu_N R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)]_\xi \simeq 0$. This is because of the elliptic factor $V_0 \subset \mathrm{char}\mathcal{M}$.

To obtain the distribution version of the above theorem, we shall use the notion of regular singularities introduced by Kashiwara-Oshima [7] and the techniques of D'Agnolo-Tonin [4]. We denote by $\mathcal{E}_X$ the sheaf of ring of micro-differential operators on T^*X.

Proposition 2.3 (Kashiwara-Oshima [7])**.** *Let $V \subset \dot{T}^*X$ be a regular involutive submanifold and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. We take a coherent $\mathcal{E}_X$-module $\mathcal{S}_V$ which is simple along V. Assume that $\mathcal{M}$ has regular singularities along V (in the sense of Kashiwara-Oshima* [7]*). Then we have an exact sequence*

$$
\mathcal{S}_V^{N_0} \longrightarrow \mathcal{M} \longrightarrow 0 \tag{2.5}
$$

of $\mathcal{E}_X$-modules for some $N_0 \geq 0$.

3. Extension theorems for distribution (C^∞-) solutions

In this section, we will give a distribution version of Theorem 2.1. Now let $\mathcal{D}b_M$ (resp. $\mathcal{A}_M$) be the sheaf of Schwartz's distributions (real analytic functions) on M and consider the exact sequence

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}b_M \underset{sp}{\longrightarrow} \dot{\pi}_* \mathcal{C}_M^f \longrightarrow 0, \tag{3.1}$$

where $\mathcal{C}_M^f$ is the sheaf of tempered microfunctions introduced by Bony [2] (see also Andronikof [1] for a functorial construction) and $\dot{\pi} : \dot{T}_M^* X \longrightarrow M$ is the projection. In the first part of the next theorem, we consider the systems introduced by D'Agnolo-Tonin [4].

Theorem 3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

(i) *We consider the same situation as in Theorem 2.1* (i). *Assume moreover:*

(a) *$V_{\mathbf{R}} = char\mathcal{M} \cap \dot{T}_M^* X$ is a smooth regular involutive submanifold of $\dot{T}_M^* X$ such that $V = char\mathcal{M}$ is a complexification of $V_{\mathbf{R}}$ in an open neighborhood of $\dot{T}_M^* X$ in T^*X.*

(b) *As a coherent $\mathcal{E}_X$-module, $\mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M} (\pi_X : T^*X \longrightarrow X)$ has regular singularities along $V = char\mathcal{M}$ on $V_{\mathbf{R}}$.*

Then we have an isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N \xrightarrow{\sim} R\Gamma_\Omega R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N. \tag{3.2}$$

(ii) *We consider the same situation as in Theorem 2.1* (ii) *and assume that the conditions* (a) *and* (b) *in* (i) *above are satisfied. Then for every proper open convex cone $U \subset \dot{T}_N^* M$ containing $\xi \in \dot{T}_N^* M$ we have the isomorphism*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N \xrightarrow{\sim} \Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N \tag{3.3}$$

for an open tuboid $\Omega \subset M$ along N satisfying the condition $U^\circ \cap \dot{T}_N M \cap C_N(\Omega) = \emptyset$.

Proof. To explain the idea of the proof, first we restrict ourselves in proving formula (3.2) in the "0-th cohomology"

$$\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N \xrightarrow{\sim} \Gamma_\Omega \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N. \tag{3.4}$$

(i) We take a section $u \in \Gamma_\Omega \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N$ and consider the extension $\tilde{u} \in \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N$ of u as a hyperfunction solution, which is possible thanks to Theorem 2.1 (i). Now we have to verify that the extension $\tilde{u}$ is again a distribution. For this purpose, we regard $\tilde{u}$ as a section of the sheaf $\mathcal{C}_M$ of microfunctions on $\dot{T}_M^* X$ which is contained in $\mathcal{C}_M^f$ on the half side $\dot{\pi}^{-1}(\Omega) \subset \dot{T}_M^* X$. It follows from the assumptions (a), (b) and Proposition 2.3 that the $\mathcal{E}_X$-module $\tilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi_X{}^{-1}\mathcal{D}_X} \pi_X{}^{-1}\mathcal{M}$ can be locally transformed to a quotient of a direct sum of the de Rham system by a real contact transformation $\Phi : U_1 \simeq U_2$ between two open subsets $U_1, U_2 \subset \dot{T}_M^* X$. We denote by $\mathcal{S}$ this de Rham system. Then we have an exact sequence $\mathcal{S}^{N_0} \longrightarrow \Phi_* \tilde{\mathcal{M}} \longrightarrow 0$ of $\mathcal{E}_X$-module for some $N_0 \geq 0$ on U_2 and the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\Phi_* \tilde{\mathcal{M}}, \mathcal{C}_M) & \longrightarrow & \mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{S}^{N_0}, \mathcal{C}_M) & \longrightarrow & \mathcal{C}_M^{N_0} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\Phi_* \tilde{\mathcal{M}}, \mathcal{C}_M^f) & \longrightarrow & \mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{S}^{N_0}, \mathcal{C}_M^f) & \longrightarrow & (\mathcal{C}_M^f)^{N_0}, \end{array} \tag{3.5}$$

in which all arrows are injective. Since the section

$$\Phi_* \tilde{u} \in \Gamma(U_2; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}^{N_0}, \mathcal{C}_M))$$

is contained in $(\mathcal{C}_M^f)^{N_0}$ on the half side $\Phi(\dot{\pi}^{-1}(\Omega) \cap U_1) \subset U_2$ of U_2 and "constant" along real bicharacteristic strips of $\mathcal{S}$ which are transversal to $\Phi(\dot{\pi}^{-1}(N) \cap U_1)$, $\Phi_* \tilde{u}$ is contained in $(\mathcal{C}_M^f)^{N_0}$ also on $\Phi(U_1) = U_2$. Therefore we get $\Phi_* \tilde{u} \in \Gamma(U_2; \mathcal{H}om_{\mathcal{E}_X}(\Phi_* \tilde{\mathcal{M}}, \mathcal{C}_M^f))$, and the invariance by quantized contact transformations of the sheaf $\mathcal{C}_M^f$ proved by Bony [2] (see also Andronikof [1]) ensures that $\tilde{u} \in \Gamma(U_1; \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M^f))$. This implies $\tilde{u} \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N$ and completes the proof of the 0-th cohomology part of (i).

The part (ii) is also proved by extending $u \in \Gamma_\Omega \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N$ as a hyperfunction solution (by Theorem 2.1 (ii)) and using the same microlocal observation as above.

Finally, we will prove formula (3.2) completely; that is, in the derived category. Since the system $\mathcal{M}$ in consideration is now hyperbolic, we may apply Corollary 6.4.4. of [9] to get the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)|_N \xrightarrow{\sim} R\Gamma_\Omega R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)|_N. \tag{3.6}$$

Hence by considering Sato's distinguished triangle, it remains to show

$$R\Gamma(U; R\dot{\pi}_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M^f)) \xrightarrow{\sim} R\Gamma(U \cap \Omega; R\dot{\pi}_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M^f)) \tag{3.7}$$

for an open neighborhood U of N. But it follows from the Mayer-Vietoris argument that the proof of the isomorphism (3.7) can be reduced to the one

$$R\Gamma(U_1; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M^f)) \xrightarrow{\sim} R\Gamma(U_1 \cap \dot{\pi}^{-1}(\Omega); R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M^f)) \tag{3.8}$$

for a sufficiently small open subset $U_1 \subset \dot{T}_M^* X$, which is equivalent to

$$R\Gamma(U_2; R\mathcal{H}om_{\mathcal{E}_X}(\Phi_* \tilde{\mathcal{M}}, \mathcal{C}_M^f)) \xrightarrow{\sim} R\Gamma(\Phi(U_1 \cap \dot{\pi}^{-1}(\Omega)); R\mathcal{H}om_{\mathcal{E}_X}(\Phi_* \tilde{\mathcal{M}}, \mathcal{C}_M^f)) \tag{3.9}$$

through the real contact transformation $\Phi : U_1 \simeq U_2$ in the proof above. If we consider a resolution of the $\mathcal{E}_X$-module $\Phi_* \tilde{\mathcal{M}}$ (by the same de Rham system $\mathcal{S}$)

$$\longrightarrow \mathcal{S}^{N_2} \longrightarrow \mathcal{S}^{N_1} \longrightarrow \mathcal{S}^{N_0} \longrightarrow \Phi_* \tilde{\mathcal{M}} \longrightarrow 0 \tag{3.10}$$

by the repeated use of Proposition 2.3, we can prove this last isomorphism (3.9) by a standard argument and the formula

$$R\Gamma(U_2; R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}, \mathcal{C}_M^f)) \xrightarrow{\sim} R\Gamma(\Phi(U_1 \cap \dot{\pi}^{-1}(\Omega)); R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}, \mathcal{C}_M^f)) \tag{3.11}$$

for the de Rham system $\mathcal{S}$. Note that the same argument was also used in D'Agnolo-Tonin [4] to solve the Cauchy problem for distributions. □

Remark 3.2. The same situation as in Theorem 3.1 (i) was first considered by D'Agnolo-Tonin [4]. In fact, we have an isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, Thom(\mathbb{C}_\Omega, \mathcal{D}b_M))|_N \tag{3.12}$$

as an immediate cosequence of their main theorem, where $Thom(*, \mathcal{D}b_M)$ is the functor TH of Kashiwara [6] and $Thom(\mathbb{C}_\Omega, \mathcal{D}b_M)$ denotes the subsheaf of $\Gamma_\Omega \mathcal{D}b_M$ consisting of sections which are extendible through N as distributions. Our Theorem 3.1 (i) removes this growth condition on the function space.

Remark 3.3. Theorem 3.1 (i) can be restated in the following way. Let $j : T^*M \longrightarrow T^*(T^*_M X) \simeq T_{(T^*_M X)}(T^*X)$ be the injection induced by the projection $T^*_M X \longrightarrow M$, where we used the Hamiltonian isomorphism $-H : T^*(T^*_M X) \simeq T_{(T^*_M X)}(T^*X)$. Then one of the main results of Kashiwara-Schapira [8] was the following estimation of the micro-support of the hyperfunction solution complexes:

$$\mathrm{SS}[R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)] \subset j^{-1}C_{(T^*_M X)}(\mathrm{char}\mathcal{M}), \tag{3.13}$$

which is valid for arbitrary coherent $\mathcal{D}_X$-module $\mathcal{M}$. Our Theorem 3.1 (i) is the distribution version of this result. That is, if we take a coherent $\mathcal{D}_X$-module $\mathcal{M}$ satisfying the conditions (a) and (b) of Theorem 3.1 (i), then we have

$$\mathrm{SS}[R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)] \subset j^{-1}C_{(T^*_M X)}(\mathrm{char}\mathcal{M}). \tag{3.14}$$

From the proof of Theorem 3.1, we also get the following Hartogs type theorem for distribution solutions, which is a refinement of the division theorem of Kashiwara [5] and Kawai [10].

Theorem 3.4. *Assume that the codimension d of N in M satisfies $d \geq 2$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is non-characteristic. We also assume that $\mathcal{M}$ satisfies the conditions* (a) *and* (b) *of Theorem 3.1* (i)*. Then we have an isomorphism*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N \xrightarrow{\sim} \Gamma_{M-N}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)|_N. \tag{3.15}$$

Remark 3.5. We can replace the sheaf $\mathcal{D}b_M$ in Theorems 3.1 and 3.4 by the sheaf $\mathcal{C}^\infty_M$ of C^∞-functions. Since Colin [3] recently proved the invariance of the sheaf of differentiable microfunctions of Bony [2] through real quantized contact transformations, the proof of Theorem 3.1 applies also to the case of C^∞-functions.

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