UNIVERSAL UNIFORM EBERLEIN COMPACT SPACES

M. BELL

(Communicated by Alan Dow)

Abstract. A universal space is one that continuously maps onto all others of its own kind and weight. We investigate when a universal Uniform Eberlein compact space exists for weight $\kappa$. If $\kappa = 2^{\omega_1}$, then they exist whereas otherwise, in many cases including $\kappa = \omega_1$, it is consistent that they do not exist. This answers (for many $\kappa$ and consistently for all $\kappa$) a question of Benyamini, Rudin and Wage of 1977.

1. Introduction

Let $\mathcal{C}$ be a class of compact Hausdorff spaces. The weight of a space $X$ is the least cardinality of a base for $X$. A universal element of $\mathcal{C}$ for weight $\kappa$ is a space $X \in \mathcal{C}$ with weight $\kappa$ such that every $Y \in \mathcal{C}$ of weight $\leq \kappa$ is a continuous image of $X$. In topology, the two most famous examples of universal elements are: the Cantor space $2^\omega$ is a universal compact space for weight $\omega$ and, under the Continuum Hypothesis, $\beta\omega \setminus \omega$ is a universal compact space for weight $\omega_1$. Regarding the second example, we mention that Shelah [Sh84], [Sh90] has shown that it is consistent that there is no universal boolean algebra (one that contains copies of all others as subalgebras) of size $\omega_1$; hence by duality, it follows that it is consistent that there is no universal compact space of weight $\omega_1$.

An Eberlein compact space (abbreviated EC) is one that is homeomorphic to a weakly compact subspace of a Banach space. A Uniform Eberlein compact space (abbreviated UEC) is one that is homeomorphic to a weakly compact subspace of a Hilbert space. EC spaces have been extensively studied by analysts and topologists; for a good introduction see the survey article of Negrepontis [Ng84]. Benyamini, Rudin and Wage [BRW77] raised the questions: for which $\kappa$ do there exist universal EC and universal UEC spaces for weight $\kappa$? Argyros and Benyamini [AB97] have shown that if $\kappa^{\omega_1} = \kappa$ or $\kappa = \omega_1$, then a universal EC for weight $\kappa$ does not exist, whereas if $\kappa$ is a strong limit of countable cofinality, then it does exist. Hence under GCH, they exist iff $\text{cf}(\kappa) = \omega$. In section 3 we prove that if $\kappa = 2^{<\kappa}$, then a universal UEC space exists for weight $\kappa$ (hence under GCH, universal UEC spaces exist for all weights $\kappa$). In section 4 we note that it follows from results of Shelah [Sh84], [Sh90] on universal graphs that it is consistent that universal UEC spaces...
Proof. Assume that \( f \) is a closed subspace of \( X \). Theorem 2.2. A boolean space is a compact Hausdorff space such that \( CO(X) \) is a basis. If \( B \) is a boolean algebra, then \( st(B) \) is the space of all ultrafilters of \( B \). The weight of \( st(B) \) is the cardinality or size of \( B \). When we say “duality”, we refer to the well-known equivalences of boolean spaces and continuous maps and boolean algebras and homomorphisms.

2. Uniform Eberleins and \( c \)-algebras

Let \( \alpha \kappa = \kappa \cup \{ \infty \} \) be the one-point compactification of the discrete space \( \kappa \) and let \( \alpha \kappa^\omega \) be the \( \omega \)th power of \( \alpha \kappa \). The following theorem is a key to our investigation.

**Theorem 2.1** (Benyamini, Rudin and Wage \[BRW\]). \( X \) is a UEC space of weight \( \leq \kappa \) iff \( X \) is a Hausdorff continuous image of a closed subspace of \( \alpha \kappa^\omega \).

So, if there is a universal UEC space for weight \( \kappa \), then there is one that is a closed subspace of \( \alpha \kappa^\omega \). Moreover, to construct a universal UEC for weight \( \kappa \), it suffices to find a closed subspace of \( \alpha \kappa^\omega \) that continuously maps onto all other closed subspaces. We decide to work in the realm of boolean algebras.

A **c-algebra** is a boolean algebra \( B \) which has a generating set \( \bigcup_{n \in \omega} B_n \) satisfying

a) each \( B_n \) consists of elements pairwise disjoint and \( B_n \cap B_m = \emptyset \) for \( n \neq m \) and

b) the **nice** property that \( \forall F \neq 1 \) for each finite \( F \subset \bigcup_{n \in \omega} B_n \). When we mention a c-algebra \( B \), we always have in mind a fixed generating set which we denote by \( \text{gen}(B) = \bigcup_{n \in \omega} B_n \) and which satisfies a) and b) and we let \( \pi : \text{gen}(B) \to \omega \) be defined by \( \pi(x) = n \) where \( x \in B_n \).

The next theorem was, in essence, communicated to the author by Petr Simon. We say that a family \( B \) of subsets of a space \( X \) is **\( T_0 \)-separating** if whenever \( x \neq y \) are points of \( X \), then there exists \( B \in B \) such that \( B \) contains exactly one of \( x, y \). By duality, a generating set \( G \) of a boolean algebra \( B \) corresponds to a \( T_0 \)-separating family of clopen subsets of \( X = st(B) \).

**Theorem 2.2.** \( X \) is homeomorphic to a closed subspace of \( \alpha \kappa^\omega \) iff \( X \) is a boolean space and \( CO(X) \) is a c-algebra of size at most \( \kappa \).

Proof. Assume that \( X \) is a closed subspace of \( \alpha \kappa^\omega \). For each \( n \) and for each \( \alpha < \kappa \), put \( B^\alpha_n = \{ f \in X : f(n) = \alpha \} \). The family \( B = \{ B^\alpha_n : n \in \omega \} \) generates \( CO(X) \). Fix a point \( x \in X \). For each \( n \) such that \( x(n) = \infty \) put \( B_{3n} = \{ B^\alpha_n : \alpha < \kappa \} \) and for each \( n \) such that \( x(n) \neq \infty \) put \( B_{3n+1} = \{ X \setminus B^\alpha_n : x(n) = \alpha \} \) and \( B_{3n+2} = \{ B^\beta_n : \beta \neq \alpha \} \). Then \( \bigcup_{n \in \omega} B_n \) generates \( CO(X) \), has the nice property, each \( B_n \) consists of elements pairwise disjoint, and we can thin each \( B_n \) to get \( \{ B_n : n < \omega \} \) a pairwise disjoint family; so, \( CO(X) \) is a c-algebra of size at most \( \kappa \).

Assume \( X \) is a boolean space and that \( CO(X) \) is a c-algebra of size at most \( \kappa \) and let \( \bigcup_{n \in \omega} B_n \) be a generating set of \( CO(X) \) witnessing this. For each \( n \), list \( B_n \) as \( \{ B^n_\alpha : \alpha \in S_n \subset \kappa \} \). Define \( \phi : X \to \alpha \kappa^\omega \) by \( \phi(x)(n) = \infty \) if \( x \notin \bigcup_{\alpha \in S_n} B^n_\alpha \).
and \( \phi(x)(n) = \alpha \) if \( x \in B^\alpha_n \). \( \phi \) is an embedding because \( \bigcup_{n<\omega} B_n \) is a \( T_0 \)-separating family.

We require some c-algebra notions. A c-algebra \( A \) is a **c-subalgebra** of \( B \), denoted by \( A < B \), if \( A \) is a boolean subalgebra of \( B \) and \( A_n \subset B_n \) for each \( n \). A function \( i : A \to B \) between two c-algebras is a **c-embedding** if \( i \) is an embedding of boolean subalgebras and \( i[A_n] \subset B_n \) for each \( n \); if, in addition, \( i[A_n] = B_n \) for each \( n \), then \( i \) is a **c-isomorphism**. \( U \) is a **universal** c-algebra of size \( \kappa \) if every c-algebra of size \( \leq \kappa \) can be c-embedded into \( U \).

A **nut** \( N \) of a c-algebra \( B \) is an element of \( B \) that is given by a description denoted by \((N^+, N^-)\) so that \( N = \bigwedge N^+ - \bigvee N^- \), where \( N^+ \) and \( N^- \) are disjoint finite subsets of \( \text{gen}(B) \). Since \( \text{gen}(B) \) generates \( B \), knowledge of precisely which nuts are 0 completely determines (up to c-isomorphism) the c-algebra \( B \). A fact that we will use in section 3 is that since a c-algebra \( B \) has the nice property, if \( N \) is a nut of \( B \) and \( N = 0 \), then \( N^+ \neq \emptyset \).

We say that \( A \) is a **closed** c-subalgebra of a c-algebra \( B \), denoted by \( A <_c B \), if \( A < B \) and

1. whenever \( N \) is a nut of \( A \) and \( F \) is a finite subset of \( \text{gen}(B) \) and \( N \leq \bigvee F \), then there exists a finite subset \( G \) of \( \text{gen}(A) \) such that \( N \leq \bigvee G \) and \( \pi[G] = \pi[F] \), and
2. whenever \( N \) is a nut of \( A \) and \( F \) is a finite subset of \( \text{gen}(B) \) and \( N \wedge \bigwedge F \neq 0 \), then there exists a finite subset \( G \) of \( \text{gen}(A) \) such that \( N \wedge \bigwedge G \neq 0 \) and \( \pi[G] = \pi[F] \).

To use closed c-subalgebras (i.e. elementarity) in our proof of Lemma 3.1 was suggested to the author by Alan Dow. Three important properties of \( A <_c B \) are:

- \( A < B \) imply that there exists \( C <_c B \) such that \( A < C \) and \( |C| = |A| \);
- \( A <_c B \) and \( A < C < B \) imply that \( A <_c C \);
- if \( A_\alpha <_c A_\beta \) for \( \alpha < \beta < \kappa \) and for each \( \alpha < \kappa \), \( A_\alpha <_c B \), then \( \bigcup_{\alpha<\kappa} A_\alpha <_c B \).

To get a universal UEC space for certain weights \( \kappa \), we will construct a universal c-algebra \( U \) of size \( \kappa \), then, by Theorem 2.2 and Theorem 2.1 \( \text{st}(U) \) will be our universal UEC. Whether this is necessary for the existence of universal UEC spaces of weight \( \kappa \) is not known. Recent research of Dzamonja [Dz98] gives sufficient conditions for the non-existence of universal c-algebras of weight \( \kappa \) (in the absence of GCH).

### 3. Existence of universal uniform Eberleins

It is convenient and economical to decree that \( \emptyset \) is a c-algebra and that for any c-algebra \( B \), \( \emptyset <_c B \) and \( \emptyset : \emptyset \to B \) is a c-embedding.

**Lemma 3.1** (Amalgamation Lemma). Let \( B \) and \( C \) be c-algebras. If \( A <_c C \) and \( i : A \to B \) is a c-embedding, then there exists a c-algebra \( E \) such that \( B < E \), \( |E| \leq |B| + |C| \), and there exists a c-embedding \( j : C \to E \) such that \( j \upharpoonright A = i \).

**Proof.** Without loss of generality we assume that \( i \) is an inclusion, so \( A < B \) (therefore, \( A_n \subset B_n \) for each \( n \)) and we assume that \( (\text{gen}(C) \setminus \text{gen}(A)) \cap B = \emptyset \). Define, for each \( n \), \( E_n = (C_n \setminus A_n) \cup B_n \). Our goal is to define a c-algebra structure \( E \) with \( \text{gen}(E) = \bigcup_{n<\omega} E_n \) which simultaneously extends the c-algebra structure of
$B$ and $C$. Formally, we will use a quotient algebra to do this. Let $D$ be the $c$-subalgebra of $C$ generated by $\bigcup_{n<\omega} (C_n \setminus A_n)$. Let $B \ast D$ be the free product of the boolean algebras $B$ and $D$, i.e., $B \ast D$ is isomorphic to $\mathrm{CO}(\text{st}(B) \times \text{st}(D))$. Let $I$ be the ideal in $B \ast D$ generated by $I_1 \cup I_2$ where $I_1 = \{(x, y) : x \in B_n \setminus A_n$ and $y \in C_n \setminus A_n, \text{ for some } n\}$ and $I_2 = \{(N, M) : N$ is a nut of $A$ and $M$ is a nut of $D$ and $N \cap M = 0 \in C\}$. Put $E = B \ast D / I$ and let $\phi : B \ast D \to B \ast D / I$ be the quotient homomorphism. $E$ is generated by $\bigcup_{n<\omega} E_n$ where $E_n = \{\phi(x, 1) : x \in B_n\} \cup \{\phi(1, y) : y \in C_n \setminus A_n\}$. $I_1$ ensures that, for each $n$, $E_n$ consists of elements pairwise disjoint.

Claim 1. $E$ is a $c$-algebra.

Proof of Claim. We must check that $E$ has the nice property. Striving for a contradiction, assume that $\bigvee_{i<n} \phi(x_i, 1) \lor \bigvee_{j<m} \phi(1, y_j) = 1$ where $x_i \in \text{gen}(B)$ and $y_j \in \text{gen}(D)$. Choose $p, q < \omega$ such that

\begin{equation}
\bigvee_{i<n} (x_i, 1) \lor \bigvee_{j<m} (1, y_j) \lor \bigvee_{k<p} (N_k, M_k) \lor \bigvee_{l<q} (z_l, w_l) = 1
\end{equation}

where each $(N_k, M_k) \in I_2$ and $(z_l, w_l) \in I_1$. Since $N_k \cap M_k = 0$ in $C$ and $C$ has the nice property, we can choose $r_k \in (N_k \cap M_k)^+ = N_k^+ \cup M_k^+$. So $(N_k, M_k) \leq (r_k, 1)$ if $r_k \in N_k^+$ or $(N_k, M_k) \leq (1, r_k)$ if $r_k \in M_k^+$. As $C$ has the nice property, $c = \bigvee_{j<m} y_j \lor \bigvee \{r_k : k < p \text{ and } r_k \in M_k^+\} \lor \bigvee_{l<q} w_l < 1$. As $B$ has the nice property, $b = \bigvee_{j<m} x_i \lor \bigvee \{r_k : k < p \text{ and } r_k \in N_k^+\} \lor \bigvee_{l<q} z_l < 1$. Thus, $(1-b, 1-c) \neq 0$ but is disjoint from the left-hand side of equation (\ref{eq:proof}).

Claim 2. $B < E$ in the form $\{\phi(x, 1) : x \in B\}$.

Proof of Claim. By virtue of the free product, if $N$ is a nut of $B$ and $N \neq 0$, then $\phi(N, 1) = 0$. We must show that if $N$ is a nut of $B$ and $N \neq 0$, then $\phi(N, 1) \neq 0$. Assume not, and let $N \neq 0$ be a nut of $B$ such that

\begin{equation}
(N, 1) \leq \bigvee_{i<n} (N_i, M_i) \lor \bigvee_{j<m} (x_j, y_j)
\end{equation}

where each $(N_i, M_i) \in I_2$ and $(x_j, y_j) \in I_1$ and $n + m$ is the least $k$ such that there exist $k$ elements of $I_1 \cup I_2$ whose join is $\geq (P, 1)$ for some nut $P$ of $B$ with $P \neq 0$. By minimality of $n + m$, we get that $N \leq \bigwedge_{i<n} N_i \land \bigwedge_{j<m} x_j$ in $B$ (otherwise, there exists $i$ such that $N - N_i \neq 0$ or $N - x_i \neq 0$ and either possibility yields a non-0 nut $P$ of $B$ such that $(P, 1)$ is $\leq$ the join of 1 less element on the right-hand side of (\ref{eq:proof}).

Also, $y = \bigvee_{i<n} M_i \lor \bigvee_{j<m} y_j = 1$ in $D$ or else $(N, 1 - y) \neq 0$ but is disjoint from the right-hand side of (\ref{eq:proof}). Since for each $i < n$, $N_i \leq M_i'$ (we use $'$ for complement) in $C$, we get that $\bigwedge_{i<n} N_i \leq \bigwedge_{i<n} M_i' \leq \bigvee_{j<m} y_j$ in $C$. As $A < C$ and $\bigwedge_{i<n} N_i$ is a nut of $A$, we can choose a finite $F \subset \text{gen}(A)$ such that $\pi[F] = \pi[\{y_j : j < m\}]$ and $\bigwedge_{i<n} N_i \leq \bigvee F$ in $A$. But, then $0 \neq N \leq \bigvee F \land \bigwedge_{j<m} x_j$ in $B$, which is a contradiction since for every $z \in F$ there exists $j < m$ such that $z$ and $y_j$ (and therefore $x_j$) are
in the same column $B_n$ of $B$ with $z \in A_n$ and $x_j \in B_n \setminus A_n$ and therefore $z \cap x_j = 0$ in $B$; hence $\bigvee F \land \bigwedge_{j<n} x_j = 0$ in $B$.

\[\text{Claim 3.} \ C < E \text{ in the form } \{\phi(x,1) : x \in A\} \cup \{\phi(1,y) : y \in D\}.\]

\[\text{Proof of Claim.} \ \text{By virtue of the free product and } I_2, \text{ if } N \text{ is a nut of } A \text{ and } M \text{ is a nut of } D \text{ and } N \land M = 0, \text{ then } \phi(N, M) = 0. \text{ We must show that if } N \text{ is a nut of } A \text{ and } M \text{ is a nut of } D \text{ and } N \land M \neq 0, \text{ then } \phi(N, M) \neq 0. \text{ Assume not and let } N \text{ be a nut of } A \text{ and let } M \text{ be a nut of } D \text{ such that } N \land M \neq 0 \text{ in } C \text{ and } (N, M) \leq \bigvee_{j<n} (N_j, M_j) : (x_j, y_j) \text{ where each } (N_j, M_j) \in I_2 \text{ and } (x_j, y_j) \in I_1.\]

Since $N \land M \neq 0$ in $C$ and for each $i < n$, $N_i \land M_i = 0$ in $C$, we get that $(N, M) - \bigvee_{i<n} (N_i, M_i) = 0$. As

\[\bigvee_{i<n} (N_i, M_i) = \bigvee_{k<p} (P_k, Q_k) \leq \bigvee_{j<m} (x_j, y_j)\]

where each $P_k$ is a nut of $A$ and $Q_k$ is a nut of $D$, we get that there exists $k < p$ with $(P_k, Q_k) \neq 0$ and $(P_k, Q_k) \leq \bigvee_{j<m} (x_j, y_j)$. So, without loss of generality we can assume that $(N, M) \leq \bigvee_{j<m} (x_j, y_j)$. As before, let $m$ be the least such that there exist $k$ elements of $I_1$ whose join is $\geq (P, Q)$ for some nut $P$ of $A$ and for some nut $Q$ of $D$ with $P \land Q \neq 0$ in $C$. By minimality of $m$, $0 \neq N \land M \leq \bigvee_{j<m} y_j$.

Therefore, $N \land \bigvee_{j<m} y_j \neq 0$. As $A < c \hspace{0.1cm} N$ is a nut of $A$, we can choose a finite $F \subset \text{gen}(A)$ such that $\pi[F] = \pi[\{y_j : j < m\}]$ and $N \land \bigvee_{j<m} y_j 
eq 0$. But, $N \leq \bigvee_{j<m} x_j$ and $\bigvee_{j<m} x_j \land \bigvee_{j<m} F = 0$ which is a contradiction.

This completes the proof of our lemma.

The proofs of the following two results are standard procedures in model theory, but due to the asymmetry of our assumptions (we mix $A < B$ and $A <_c B$) we were unable to find a theorem to quote, and so we present the proofs.

\[\text{Proposition 3.2.} \ \text{Let } \lambda = \lambda^c < \kappa \text{ and let } B \text{ be a } c \text{-algebra of size } \kappa. \text{ Then there exists a } c \text{-algebra } D \text{ of size } \lambda \text{ such that } B < D \text{ and such that every } c \text{-embedding of any } c \text{-algebra } A \text{ of size } \kappa \text{ into } D \text{ can be extended to a } c \text{-embedding of } C, \text{ if } C \text{ is any } c \text{-algebra of size } < \kappa \text{ with } A <_c C.\]

\[\text{Proof.} \ \text{List all triples } (X, Y, i) \text{ where } Y \text{ is a } c \text{-algebra with } Y_n \subset \kappa \times \{n\} \text{ for each } n, \ X <_c Y \text{ and } i \text{ is an injection of } X \text{ into } \lambda \times \omega \text{ as } (X_\alpha, Y_\alpha, i_\alpha)_{\alpha < \lambda} \text{ so that for each } \alpha, i_\alpha[X_\alpha] \subset \alpha \times \omega. \text{ In our listing we allow the degenerate cases where } X = i = \emptyset \text{ in order to start our embeddings. We can do this listing because } \lambda = \lambda^c < \kappa.\]

We will build a $c$-algebra structure $D$ with $\text{gen}(D) = \lambda \times \omega$ by recursively constructing $D_\alpha$ with $\text{gen}(D_\alpha) = \lambda_\alpha \times \omega$ such that for $\alpha < \beta < \lambda$, $\alpha \leq \lambda_\alpha \leq \lambda_\beta$ and $D_\alpha < D_\beta$. To begin, let $\lambda_0 = |B|$ and make $D_0$ $c$-isomorphic to $B$. If $\alpha$ is a limit, put $\lambda_\alpha = \sup \{\lambda_\beta : \beta < \alpha\}$ and $D_\alpha = \bigsqcup_{\beta < \alpha} D_\beta$. If $\alpha = \beta + 1$ is a successor, then we have $D_\beta$ with $\text{gen}(D_\beta) = \lambda_\beta \times \omega$. If $i_\beta$ is not a $c$-embedding of $X_\beta$ into $D_\beta$, then put $\lambda_\beta = \lambda_\beta$ and $D_\alpha = D_\beta$. If $i_\beta$ is a $c$-embedding of $X_\beta$ into $D_\beta$, then apply the Amalgamation Lemma $[\text{K}]$ with $A = X_\beta, B = D_\beta, C = Y_\beta$ and $i = i_\beta$ to yield
a c-algebra $E$ such that $D_\beta < E$, $|E| \leq |D_\beta| + |Y_\beta|$ and $i_\beta$ extends to $Y_\beta$. Put $D_\alpha = E$ and $\lambda_\alpha$ is the first ordinal $> \lambda_\beta$ that can accommodate the extra elements of gen$(E)$.

Now, let $D = \bigcup_{\alpha < \lambda} D_\alpha$. If $A <_c C$, $|C| < \kappa$ and $i : A \to D$ is a c-embedding, then we can identify $(A, C, i)$ with a $(X_\beta, Y_\beta, i_\beta)$ for some $\beta < \lambda$ and at stage $\alpha = \beta + 1$, we extended $i$ to $Y_\beta = C$.

Theorem 4.1. Let Theorem 4.1. into a graph $G$ on a set $X$ such that for all $x,y \in Y$, $\{x,y\} \in H \iff \{i(x),i(y)\} \in G$. A universal graph of size $\kappa$ is a graph $G$ on a set $X$ of size $\kappa$ such that every graph on any set of size $\leq \kappa$ embeds into $G$.

Theorem 4.2. If $\kappa = 2^{<\kappa}$, then there exists a universal c-algebra $U$ of size $\kappa$.

Proof. If $\kappa = \kappa^{<\kappa}$, then let $U$ be the $D$ of the previous proposition with $\lambda = \kappa$ and $B = \emptyset$.

Otherwise, $\kappa$ is a singular strong limit cardinal, so let $\kappa = \sup\{\kappa_\alpha : \alpha < \text{cf}(\kappa)\}$ where $\alpha < \beta$ implies $\kappa_\alpha < \kappa_\beta$, $\alpha$ a limit implies $\kappa_\alpha = \sup\{\kappa_\beta : \beta < \alpha\}$, and $\kappa_{\alpha+1} = 2^{\kappa_\alpha}$. By induction on $\alpha < \text{cf}(\kappa)$, we construct c-algebras $U_\alpha$ with, for each $\alpha$, $U_\alpha \subseteq \kappa_\alpha \times \{n\}$ such that $\alpha < \beta$ implies $U_\alpha < U_\beta$ and $\alpha$ a limit implies $U_\alpha = \bigcup_{\beta < \alpha} U_\beta$. For $\alpha = \beta + 1$, we apply the preceding proposition with $B = U_\beta$ and put $U_\alpha$ as the $D$ of that proposition. Now, let $U = \bigcup_{\alpha < \text{cf}(\kappa)} U_\alpha$.

In either case, to show that our $U$ is universal, let $A$ be a c-algebra of size $\leq \kappa$. Write $A$ as $\bigcup_{\alpha < \text{cf}(\kappa)} A_\alpha$ where for each $\alpha$, $A_\alpha <_c A$, $|A_\alpha| < \kappa$, $\alpha < \beta$ implies $A_\alpha < A_\beta$ and $\alpha$ a limit implies $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$. We then have that for $\beta < \alpha$, $A_\beta <_c A_\alpha$. Finally, an induction up to $\text{cf}(\kappa)$, using the preceding proposition, will embed $A$ into $U$.

4. Non-existence of Universal Uniform Eberleins

$G \subseteq [X]^2$ (the doubletons of $X$) is called a graph on $X$. A graph $H$ on a subset $Y$ of $X$ is called a subgraph of $G$ if $H = G \cap [Y]^2$. A graph $H$ on a set $Y$ embeds into a graph $G$ on a set $X$ if there exists a function $i : Y \to X$ such that for all $x,y \in Y$, $\{x,y\} \in H \iff \{i(x),i(y)\} \in G$. A universal graph of size $\kappa$ is a graph $G$ on a set $X$ of size $\kappa$ such that every graph on any set of size $\leq \kappa$ embeds into $G$.

Theorem 4.2. Let $\mathcal{C}$ be a class of compact Hausdorff spaces containing all closed subspaces of $\kappa \times \kappa$. Then, if there exists a universal element in $\mathcal{C}$ for weight $\kappa$, there exists a universal graph of size $\kappa$.

Proof. Let $X$ be a universal element in $\mathcal{C}$ for weight $\kappa$. Let $G$ be the intersection graph on $CO(X)$, i.e., for $x \neq y$ in $CO(X)$, $\{x,y\} \in G$ iff $x \cap y = \emptyset$. We claim that $G$ is a universal graph of size $\kappa$. To prove this, let $H$ be any graph on an $S \subseteq \kappa$. Consider the following closed subspace of $\kappa \times \kappa$, $Y = \kappa \times \{\infty\} \cup \{\infty\} \times \kappa \cup \{(\alpha,\beta),(\beta,\alpha) : \{\alpha,\beta\} \in H\}$. Let $\phi : X \to Y$ be a continuous surjection. For each $\alpha < \kappa$, put $B_\alpha = \{\alpha \times \{\alpha\} \cup \{\alpha\} \times \kappa\} \cap Y$. For each $\alpha < \kappa$, $B_\alpha \subseteq CO(Y)$ and $B_\alpha \cap B_\beta \neq \emptyset$ iff $\{\alpha,\beta\} \in H$. So, the mapping $i : S \to CO(X)$ defined by $i(\alpha) = \phi^{-1}[B_\alpha]$ embeds $H$ into $G$.

Corollary 4.2. If $V$ is a model with $\lambda^{<\lambda} = \lambda < \kappa < \mu$ and $P$ is a Cohen forcing that adds $\mu$ Cohen subsets of $\lambda$, then in the model $V^P$ there is no universal UEC in the weight $\kappa$ for any $\kappa$ with $\lambda < \kappa < \mu$. In particular, if we add $\omega_2$ Cohen reals to $V$, then there is no universal UEC in the weight $\omega_1$.  

Proof. Shelah [Sh90] has shown that there is no universal graph of size $\kappa$ in this model.

An additional result, just following from cardinal arithmetic, is the result of Kojman and Shelah [KS92], that if $\text{cf}(2^\lambda) \leq \kappa < 2^\lambda$, then there does not exist a universal graph of size $\kappa$; hence, there would be no universal UEC for this weight $\kappa$.

In conclusion, we mention one open problem: Does a universal UEC for weight $\kappa$ not exist if $\kappa$ is a singular cardinal which is not a strong limit?

References


Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

E-mail address: mbell@cc.umanitoba.ca

URL: http://home.cc.umanitoba.ca/~mbell/