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C_0-SEMIGROUPS GENERATED BY SECOND ORDER DIFFERENTIAL OPERATORS WITH GENERAL WENTZELL BOUNDARY CONDITIONS

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Abstract. Let us consider the operator $\widetilde{A}u(x) = \phi(x, u'(x))u''(x)$, where ϕ is positive and continuous in $(0,1) \times \mathbf{R}$ and $\widetilde{A}$ is equipped with the so-called generalized Wentzell boundary condition which is of the form $a\widetilde{A}u+bu'+cu = 0$ at each boundary point, where $(a,b,c) \neq (0,0,0)$. This class of boundary conditions strictly includes Dirichlet, Neumann and Robin conditions.

Under suitable assumptions on ϕ, we prove that $\widetilde{A}$ generates a positive C_0-semigroup on $C[0,1]$ and, hence, many previous (linear or nonlinear) results are extended substantially.

Introduction

In analysis, the boundary conditions associated with a second order (linear or nonlinear) elliptic differential operator usually involve the function and its first derivative (including Dirichlet, Neumann and Robin conditions). In Markov process theory, following the work of A.D. Wentzell [9], it was recognized that it is natural to include boundary conditions involving the operator itself.

Thus, if A denotes an elliptic operator, the elliptic problem

$$Au(x) - \lambda u(x) = h(x)$$

or the parabolic problem

$$\frac{\partial u}{\partial t} - Au = 0$$

for $x \in \Omega \subset \mathbf{R}^n$, $t \geq 0$, is said to be equipped with the *Wentzell boundary condition* if one demands $Au(x) = 0$ for $x \in \partial\Omega$ (and any $t \geq 0$ in the parabolic case). The usual boundary condition can be written as

$$b\frac{\partial u}{\partial n} + cu = 0 \qquad \text{on} \qquad \partial\Omega,$$

where $\dfrac{\partial u}{\partial n}$ is the outer normal derivative and $(b,c) \neq (0,0)$.

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A more general boundary condition which arises naturally in this context is the *generalized Wentzell boundary condition* (GWBC)

$$aAu(x) + b\frac{\partial u}{\partial n}(x) + cu(x) = 0, \qquad x \in \partial\Omega,$$

where $(a, b, c) \neq (0, 0, 0)$. (Note that (a, b, c) can depend on x as well.)

This paper will be devoted to a rigorous analytic treatment of these ideas in the one space dimensional context of semigroups generated by (linear or nonlinear) second order operators acting in the space $C[0, 1]$.

§1. Linear case: Generation

Let $X := C[0, 1]$, the space of all real-valued continuous functions on $[0, 1]$ under the supremum norm. Suppose $\alpha : (0, 1) \to \mathbf{R}$ and $a = (a_0, a_1)$, $b = (b_0, b_1)$, $c = (c_0, c_1)$ such that

$$(H_1) \qquad \alpha \in C(0,1),\ \alpha > 0 \quad \text{on} \quad (0,1), \quad \frac{1}{\alpha} \in L^1(0,1),$$

$$(H_2) \qquad \begin{cases} (a_j, b_j, c_j) \neq (0,0,0) & \text{and} \quad a_j \geq 0, a_0 + a_1 > 0, \\ (a_j > 0) \quad \text{implies} & ((-1)^j b_j \leq 0, c_j \geq 0 \quad \text{and} \\ & c_j = 0 \quad \text{if} \quad b_j = 0), \\ (a_j = 0) \quad \text{implies} & ((-1)^j b_j < 0 \quad \text{and} \quad c_j > 0), \end{cases}$$

$$(H_3) \qquad (a_1 b_0 \neq a_0 b_1) \quad \text{or} \quad (b_1 c_0 \neq b_0 c_1).$$

Let us introduce the operator $Au := \alpha u''$ defined in the domain

$$D_{abc}(A) := \{u \in X \cap C^2(0,1) |\, Au \in X, a_j Au(j) + b_j u'(j) + c_j u(j) = 0, j = 0, 1\}$$

and notice that assumption (H_1) yields that every $u \in X \cap C^2(0, 1)$ with $\alpha u'' \in L^\infty(0, 1)$ has $u'' \in L^1(0, 1)$ and belongs to $C^1[0, 1]$. Indeed, from

$$|u''(x)| = |\frac{\alpha(x)u''(x)}{\alpha(x)}| \leq \frac{\|Au\|_\infty}{\alpha(x)}$$

it follows that $u'' \in L^1(0, 1)$ and

$$\|u''\|_{L^1} \leq \|Au\|_\infty \|\frac{1}{\alpha}\|_{L^1} < \infty.$$

Hence $u \in C^1[0, 1]$ and $u' \in AC[0, 1]$.

Theorem 1.1. *Under assumptions $(H_1)-(H_3)$, the operator $(A, D_{abc}(A))$ is densely defined, m-dissipative on X (i.e. it is dissipative and for all $\lambda > 0$ the range of $I - \lambda A$, $R(I - \lambda A)$ is all of X) and satisfies the positive minimum principle (i.e. if $0 \leq f \in D_{abc}(A)$, $x_0 \in [0, 1]$, $f(x_0) = 0$, then $Af(x_0) \geq 0$).*

The proof of this theorem will be given in Section 2, in the more general setting of the nonlinear case.

Remark 1.2. Theorem 1.1 is still true if the hypothesis $a_0 + a_1 > 0$ is omitted from (H_2), but if $a_0 = a_1 = 0$, the boundary condition reduces to (dissipative) Robin type condition

$$(-1)^{j+1} b_j u'(j) + c_j u(j) = 0.$$

Thus the hypothesis $a_0 + a_1 > 0$ means that the second order term Au plays a role in the boundary conditions in at least *one* endpoint.

Moreover, if $a_j > 0$ and $b_j = 0 = c_j$ for $j = 0, 1$, then (GWBC) reduces to the usual Wentzell boundary condition.

We observe also that assumption (H_2) can be compared with the *transversal Wentzell boundary condition* for Feller semigroups on bounded domains of $\mathbf{R}^n$, as defined in [8, p.31].

Finally, we remark that if for some j, $a_j = 0$, $b_j = 0$ and $c_j > 0$, then A will be m-dissipative but *not* densely defined; functions in $\overline{D(A)}$ will necessarily vanish at $x = j$. This will lead to a C_0- contraction semigroup on a suitable proper subspace of X.

§2. Nonlinear case: Generation

Let us introduce a function of two variables

$$\phi : (0,1) \times \mathbf{R} \to \mathbf{R}$$

which satisfies the following assumptions:

(H_4) $\phi \in C((0,1) \times \mathbf{R})$ and for all $x \in (0,1)$, $\phi(x, \cdot) \in C^1(\mathbf{R})$;

(H_5) there exists $\alpha \in C(0,1)$, $\dfrac{1}{\alpha} \in L^1(0,1)$ such that for all $(x, \xi) \in (0,1) \times \mathbf{R} : \phi(x, \xi) \geq \alpha(x) > 0$.

If $a, b, c \in \mathbf{R}^2$, $a = (a_0, a_1)$, $b = (b_0, b_1), c = (c_0, c_1)$ verify (H_2)–(H_3), let us define $(\widetilde{A}, D_{abc}(\widetilde{A}))$ as follows:

$$D_{abc}(\widetilde{A}) := \{u \in X \cap C^2(0,1) |\, \widetilde{A}u \in X,\ a_j(\widetilde{A}u)(j) + b_j u'(j) + c_j u(j) = 0,\ j = 0, 1\},$$
$$\widetilde{A}u(x) := \phi(x, u'(x))u''(x), \quad u \in D_{abc}(\widetilde{A}),\ x \in (0,1)$$

and saying $\widetilde{A}u \in X$ means $\lim_{x \to j} \widetilde{A}u(x)$ exists for $j = 0, 1$.

Remark 2.1. Note that $(\widetilde{A}, D_{abc}(\widetilde{A}))$ becomes $(A, D_{abc}(A))$ when $\phi(x, \xi) \equiv \alpha(x)$ is independent of ξ. Moreover, both the operator $\widetilde{A}$ and the boundary condition at $x = j$ (where $c_j > 0$) are nonlinear, unless ϕ is independent of ξ.

Theorem 2.2. *Under assumptions (H_1)–(H_5), the operator $(\widetilde{A}, D_{abc}(\widetilde{A}))$ is closed, densely defined and m-dissipative on X.*

Proof. First let us remark that $u \in D_{abc}(\widetilde{A})$ implies $\phi u'' \in C[0,1]$, whence, pointwise on $(0,1)$, we have

$$|u''(x)| = \frac{|\phi(x, u'(x))u''(x)|}{\phi(x, u'(x))} \leq \frac{\|\widetilde{A}u\|_\infty}{\alpha(x)}.$$

Hence

$$\|u''\|_{L^1} \leq \|\widetilde{A}u\|_\infty \|\frac{1}{\alpha}\|_1 < \infty. \tag{2.1}$$

Thus $u \in C^1[0,1]$ and $u' \in AC[0,1]$. In particular, $u'(j)$ makes sense for $j = 0, 1$. The proof will consist of four steps.

Step 1. $\widetilde{A}$ is dissipative on X.

According to the definition of dissipativity, we have to show that for all $u_1, u_2 \in D_{abc}(\widetilde{A})$ there is $f \in \mathcal{J}(u_1 - u_2)$ such that $\langle \widetilde{A}u_1 - \widetilde{A}u_2, f \rangle \leq 0$, where $\mathcal{J}$ is the duality map defined, for each $u \in X$, by

$$\mathcal{J}(u) := \{f \in X^* |\, \langle u, f \rangle = \|u\|^2 = \|f\|^2\}.$$

Let u_1, u_2 be distinct vectors in $D_{abc}(\widetilde{A})$ and assume without loss of generality that $\|u_1 - u_2\| > 0$. Let $u := u_1 - u_2$ and choose $x_0 \in [0,1]$ such that $u(x_0) = \pm\|u\|$. By interchanging u_1 and u_2, if necessary, we may suppose $u(x_0) = \|u\|$.

If $0 < x_0 < 1$, then by the first and second derivative tests of calculus,

$$(u_1 - u_2)'(x_0) = 0,\ (u_1 - u_2)''(x_0) \le 0.$$

Hence

$$u_1'(x_0) = u_2'(x_0),\ u_1''(x_0) \le u_2''(x_0). \tag{2.2}$$

Since $\delta_{x_0}\|u\|_\infty \in J(u)$, it suffices to show

$$\langle \widetilde{A}u_1 - \widetilde{A}u_2, \delta_{x_0}\rangle = \widetilde{A}u_1(x_0) - \widetilde{A}u_2(x_0) \le 0.$$

Indeed, as a consequence of (2.2) and (H_5), we have

$$\begin{aligned}(\widetilde{A}u_1 - \widetilde{A}u_2)(x_0) &= \phi(x_0, u_1'(x_0))u_1''(x_0) - \phi(x_0, u_2'(x_0))u_2''(x_0)\\ &= \phi(x_0, u_1'(x_0))(u_1''(x_0) - u_2''(x_0)) \le 0.\end{aligned}$$

Now, let us assume that $x_0 = 0$. Clearly $u'(0) \le 0$ and

$$a_0(\widetilde{A}u_1(0) - \widetilde{A}u_2(0)) + b_0u'(0) + c_0u(0) = 0.$$

If $a_0 > 0$, then according to (H_2) we have $b_0 \le 0$ and $c_0 \ge 0$ and hence

$$\widetilde{A}u_1(0) - \widetilde{A}u_2(0) = \frac{1}{a_0}(-b_0u'(0) - c_0u(0)) \le 0.$$

The case $a_0 = 0$ cannot happen because, otherwise, from (H_2) we should deduce that $b_0 < 0$ and $c_0 > 0$ and consequently

$$b_0u'(0) + c_0u(0) > 0,$$

in contradiction with the assumed boundary condition.

The case $x_0 = 1$ can be treated in the same way, taking into account that $u'(1) \ge 0$.

Step 2. $\widetilde{A}$ is closed.

Let $(u_n)_{n\in\mathbf{N}}$ be a sequence in $D_{abc}(\widetilde{A})$ with

$$\lim_{n\to\infty} u_n = u \in X$$

and

$$\lim_{n\to\infty} \widetilde{A}u_n = v \in X.$$

Then

$$v(x) = \lim_{n\to\infty} \widetilde{A}u_n(x) = \lim_{n\to\infty} \phi(x, u_n'(x))u_n''(x)$$

uniformly in $[0,1]$. On the other hand, since for arbitrary $\epsilon > 0$ and suitable $\delta > 0$ we have

$$\phi(x,\xi) \ge \delta > 0$$

on $[\epsilon, 1-\epsilon] \times \mathbf{R}$, it follows that $(u_n)_{n\in\mathbf{N}}$ is bounded in $C^2[\epsilon, 1-\epsilon]$ and relatively compact in $C^1[\epsilon, 1-\epsilon]$, i.e. $(u_n)_{n\in\mathbf{N}}$ is relatively compact in $C^1_{loc}(0,1)$. Thus, on

$(0,1)$ it follows from $\lim_{n\to\infty}\phi(x,u_n'(x))u_n''(x)=v(x)$ that $v\in C_{loc}(0,1)$ and since $u_n\to u$ in $C^2_{loc}(0,1)$ we deduce that

$$v(x)=\phi(x,u'(x))u''(x),\qquad x\in(0,1). \tag{2.3}$$

Thus, from

$$\|v\|_\infty\geq|\phi(x,u')u''(x)|\geq\alpha(x)|u''(x)|,$$

we have that

$$|u''(x)|\leq\frac{\|v\|_\infty}{\alpha(x)}$$

and hence $u''\in L^1(0,1)$. Therefore

$$u'(x)-u'(0)=\int_0^x u''(y)\,dy$$

gives that $u\in C^1[0,1]$ and $u'\in AC[0,1]$. Finally, taking into account (2.3), we conclude that

$$\lim_{x\to j}\phi(x,u'(x))u''(x)=v(j)$$

and thus $u\in D_{abc}(\widetilde{A})$ and $\widetilde{A}u=v$.

Step 3. $\widetilde{A}$ is densely defined.

Let $\epsilon>0$ be given and pick $f\in C^2[0,1]$. We seek a function $\omega\in D_{abc}(\widetilde{A})$ approximating f. We define

$$u(j)=f(j)\qquad\text{if}\qquad j=0,1,$$

and take

$$u'(j)=\begin{cases}-\frac{c_j}{b_j}f(j) & \text{if}\quad a_j\geq 0,\ b_j\neq 0,\\ 0 & \text{if}\quad a_j>0,\ b_j=0.\end{cases}$$

Remark that by (H_2), if $a_j>0$ and $b_j=0$, then $c_j=0$, whence if u is linear on some interval containing j, then

$$\widetilde{A}u(j)=\lim_{x\to j}\phi(x,u'(x))u''(x)=0,$$

so that the boundary condition

$$\widetilde{A}u(j)=-\frac{c_j}{a_j}f(j)$$

holds (notice that $c_j=0$).

Next we extend u to be a linear function on the two intervals $[0,\delta]$ and $[1-\delta,1]$ with $\delta>0$ sufficiently small, so that u satisfies the boundary condition as in $D_{abc}(\widetilde{A})$. Moreover by taking δ small enough, we have

$$\|f-u\|_{C([0,\delta]\cup[1-\delta,1])}<\frac{\epsilon}{6}.$$

In addition, on the interval $[\frac{\delta}{2},1-\frac{\delta}{2}]$ we can construct $u_1\in C^\infty[0,1]$ such that

$$\|u_1-f\|_{C[\frac{\delta}{2},1-\frac{\delta}{2}]}<\frac{\epsilon}{6}.$$

Fix $0<\epsilon_1\leq\frac{\delta}{2}$ and take $\theta\in C^\infty[\frac{\delta}{2},\frac{\delta}{2}+\epsilon_1]$ with

$$0\leq\theta(x)\leq 1,\ \theta(\frac{\delta}{2})=1,\quad \theta(\frac{\delta}{2}+\epsilon_1)=0$$

and

$$\theta^{(j)}(\frac{\delta}{2}) = \theta^{(j)}(\frac{\delta}{2} + \epsilon_1) = 0, \qquad j = 1, 2.$$

Introduce the function h defined by

$$h(x) = \theta(x)u(x) + (1 - \theta(x))u_1(x)$$

and observe that $h \in C^2[\frac{\delta}{2}, \frac{\delta}{2} + \epsilon_1]$. Hence

$$h(\frac{\delta}{2}) = u(\frac{\delta}{2}), \quad h(\frac{\delta}{2} + \epsilon_1) = u_1(\frac{\delta}{2} + \epsilon_1).$$

In addition, since

$$\begin{aligned} h'(x) &= \theta'(x)u(x) + \theta(x)u'(x) - \theta'(x)u_1(x) + (1 - \theta(x))u_1'(x), \\ h''(x) &= \theta''(x)u(x) + 2\theta'(x)u'(x) + \theta(x)u''(x) \\ &\quad - \theta''(x)u_1(x) - 2\theta'(x)u_1'(x) + (1 - \theta(x))u_1''(x), \end{aligned}$$

we obtain

$$\begin{aligned} h'(\frac{\delta}{2}) &= u'(\frac{\delta}{2}), \qquad h''(\frac{\delta}{2}) = u''(\frac{\delta}{2}) = 0, \\ h'(\frac{\delta}{2} + \epsilon_1) &= u_1'(\frac{\delta}{2} + \epsilon_1) \\ h''(\frac{\delta}{2} + \epsilon_1) &= u_1''(\frac{\delta}{2} + \epsilon_1). \end{aligned}$$

Moreover, for any $x \in [\frac{\delta}{2}, \frac{\delta}{2} + \epsilon_1]$ we have

$$\begin{aligned} |h(x) - f(x)| &= |h(x) - (1 - \theta(x))f(x) - \theta(x)f(x)| \\ &= |\theta(x)[u(x) - f(x)] + (1 - \theta(x))[u_1(x) - f(x)]| \\ &\le |u(x) - f(x)| + |u_1(x) - f(x)| \\ &\le \frac{\epsilon}{3}. \end{aligned}$$

Define

$$\omega(x) = \begin{cases} u(x), & 0 \le x \le \frac{\delta}{2}, 1 - \frac{\delta}{2} \le x \le 1, \\ h(x), & \frac{\delta}{2} \le x \le \frac{\delta}{2} + \epsilon_1, \\ u_1(x), & \frac{\delta}{2} + \epsilon_1 \le x \le 1 - \frac{\delta}{2} - \epsilon_1, \\ h_1(x), & 1 - \frac{\delta}{2} - \epsilon_1 \le x \le 1 - \frac{\delta}{2}, \end{cases}$$

where the function h_1 is defined on the interval $[1 - \frac{\delta}{2} - \epsilon_1, 1 - \frac{\delta}{2}]$ in analogy with the function h, provided that θ is replaced by $\theta_1 \in C^\infty[1 - \frac{\delta}{2} - \epsilon_1, 1 - \frac{\delta}{2}]$ where

$$0 \le \theta_1(x) \le 1, \quad \theta_1(1 - \frac{\delta}{2} - \epsilon_1) = 0, \quad \theta_1(1 - \frac{\delta}{2}) = 1$$

and

$$\theta_1^{(j)}(1 - \frac{\delta}{2} - \epsilon_1) = \theta_1^{(j)}(1 - \frac{\delta}{2}) = 0, \qquad j = 1, 2.$$

Then, $\omega \in D_{abc}(\widetilde{A})$ and obviously

$$\|\omega - f\|_{C[0,1]} \le \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.$$

Thus any function in $C^2[0,1]$ can be uniformly approximated by functions in $D_{abc}(\widetilde{A})$. Since $C^2[0,1]$ is dense in $C[0,1]$, the density of $D_{abc}(\widetilde{A})$ in $C[0,1]$ follows.

Step 4. $R(I-\lambda\widetilde{A}) = X$ for some $\lambda > 0$.

We show that $R(I-\lambda\widetilde{A})$ is dense in X for some $\lambda > 0$. Indeed, this will imply that $\widetilde{A}$ is m-dissipative, since $\widetilde{A}$ is already closed.

Let $f \in C^2[0,1]$, $\lambda > 0$. We want to solve

$$u - \lambda\widetilde{A}u = f \tag{2.4}$$

for a particular choice of $\lambda > 0$ and for all $f \in C^2[0,1]$. Fix any $\lambda > 0$ and consider $g \in C^2[0,1]$ such that g is linear on $[0,\delta]$ and on $[1-\delta,1]$, with $\delta > 0$ and δ small enough. Step 3 showed how to construct such a dense set in $C[0,1]$ of functions g of this form. Hence, it suffices to find $u \in D_{abc}(\widetilde{A})$ such that

$$u - \lambda\widetilde{A}u = g \tag{2.5}$$

From the equations

$$u(j) - \lambda\widetilde{A}u(j) = g(j),$$
$$a_j\widetilde{A}u(j) + b_ju'(j) + c_ju(j) = 0,$$

we obtain that

$$a_j(\frac{u(j)-g(j)}{\lambda}) + b_ju'(j) + c_ju(j) = 0$$

or, equivalently,

$$(\frac{a_j}{\lambda} + c_j)u(j) + b_ju'(j) = \frac{a_j}{\lambda}g(j) \tag{2.6}$$

for $j = 0,1$. Since by hypothesis (H_3) the determinant

$$\begin{aligned}\begin{vmatrix} \frac{a_0}{\lambda} + c_0 & b_0 \\ \frac{a_1}{\lambda} + c_1 & b_1 \end{vmatrix} &= \frac{a_0b_1}{\lambda} + b_1c_0 - b_0c_1 - \frac{b_0a_1}{\lambda} \\ &= \frac{a_0b_1 - b_0a_1}{\lambda} + (b_1c_0 - b_0c_1) \neq 0,\end{aligned}$$

there exists a unique solution v of the system $v'' = 0$ and (2.6). Let us consider now the equation

$$w - \lambda\phi(x, w'(x) + v')w'' = g - v \tag{2.7}$$

(note that v' is constant) with the boundary conditions

$$(\frac{a_j}{\lambda} + c_j)w(j) + b_jw'(j) = 0, \qquad j = 0,1, \tag{2.8}$$

and apply similar arguments to those used in [6] in order to find a solution $w \in C[0,1] \cap C^2(0,1)$ which satisfies (2.7) and (2.8). Then, defining $u = w + v$, the function $u \in D_{abc}(\widetilde{A})$ and satisfies (2.5).

Step 5. $\widetilde{A}$ satisfies the positive minimum principle.

In order to prove that $(\widetilde{A}, D_{abc}(\widetilde{A}))$ satisfies the positive minimum principle, we have to show that

$$0 \leq f \in D_{abc}(\widetilde{A}),\ x_0 \in [0,1],\ f(x_0) = 0 \implies \widetilde{A}f(x_0) \geq 0.$$

Indeed, if $x_0 \in]0,1[$, then $f'(x_0) = 0$ and $f''(x_0) \geq 0$, whence $\widetilde{A}f(x_0) \geq 0$.

Assume now $x_0 = 0$ and suppose by contradiction that $\widetilde{A}f(0) < 0$, that is, $\phi(0, f'(0))f''(0) < 0$.

Hence, there exists $\delta > 0$ such that

$$\phi(x, f'(x))f''(x) > 0 \qquad \text{and} \qquad \phi(x, f'(x)) > 0$$

for all $0 < x < \delta$. Consequently, $f''(x) < 0$ in $]0, \delta[$ and f' is strictly decreasing in $]0, \delta[$.

Let us consider the boundary condition at 0, which reduces to

$$a_0\widetilde{A}f(0) + b_0 f'(0) = 0$$

and distinguish two cases : $a_0 = 0$ and $a_0 > 0$.

If $a_0 = 0$, then according to (H_2), we have $b_0 < 0$ and we deduce $f'(0) = 0$. Since f' is strictly decreasing in $]0, \delta[$, it follows that $f'(x) < 0$, for all $x \in]0, \delta[$. This implies $f(x) < 0$ in $]0, \delta[$, contrary to the assumption $f \geq 0$ in $[0, 1]$.

If $a_0 > 0$, then $b_0 < 0$ (otherwise $\widetilde{A}f(0) = 0$) and

$$f'(0) = -\frac{a_0}{b_0}\widetilde{A}f(0) < 0.$$

Then, again, $f'(x) < 0$ in $]0, \delta[$ and we contradict the positivity assumption on f.

If $x_0 = 1$, analogous arguments as in the previous case still work. □

Remark 2.3. In [5] Goldstein and Lin assumed ϕ on $C([0, 1] \times \mathbf{R})$, but they never used the continuity of ϕ at $x = 0, 1$.

So, for the case of $a_0 = a_1 = 0$, the statement of Theorem 2.2 contains cases not covered in the statement of Goldstein and Lin (but covered by the proof in [5, Theorem 1]).

Moreover, in the context of Theorem 1.1 or 2.1, when for $j = 0, 1$ we have $(a_j, b_j, c_j) = (1, 0, 0)$, the restriction that $\frac{1}{\alpha} \in L^1(0, 1)$ is not necessary, as proved by Goldstein and Lin in [5, Theorem 1]. It becomes important when we want to allow $(b_j, c_j) \neq (0, 0)$.

We remark that Theorem 1.1 extends a part of Clément and Timmermanns' theorem (see [1, Proposition 1]), while Theorem 2.2 fully extends the result of Goldstein and Lin in [4].

Note that our main relevant form is when $a_j > 0$ and $(b_j, c_j) \neq (0, 0)$.

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