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SPECTRAL STRUCTURE AND SUBDECOMPOSABILITY OF p-HYPONORMAL OPERATORS

RUAN YINGBIN AND YAN ZIKUN

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Abstract. We prove that for every p-hyponormal operator $A, 0 < p \leq 1$, there corresponds a hyponormal operator $\tilde{A}$ such that A and $\tilde{A}$ have "equal spectral structure". We also prove that every p-hyponormal operator $A, 0 < p \leq 1$, is subdecomposable. Then some relevant quasisimilarity results are obtained, including that two quasisimilar p-hyponormal operators have equal essential spectra.

1. Introduction and notation

Let $\mathbf{H}$ be a complex separable Hilbert space and let $L(\mathbf{H})$ denote the algebra of all bounded linear operators on $\mathbf{H}$. An operator $A \in L(\mathbf{H})$ is said to be p-hyponormal, $0 < p \leq 1$, denoted as $A \in p\text{-}\mathbf{H}$, if $(AA^*)^p \leq (A^*A)^p$. An 1-hyponormal operator is hyponormal, and a $\frac{1}{2}$-hyponormal operator is said to be semi-hyponormal. In the sequel, for every $A \in L(\mathbf{H})$, we define $\hat{A}$ by $\hat{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ where $U, |A|$ are as in the polar decomposition $A = U|A|$. Let $\hat{A}$ have the polar decomposition $\hat{A} = V|\hat{A}|$. The operator $\tilde{A}$ is then defined by $\tilde{A} = |\hat{A}|^{\frac{1}{2}}V|\hat{A}|^{\frac{1}{2}}$. Aluthge [1] showed that for $A \in p\text{-}\mathbf{H}, 0 < p \leq 1, \hat{A}$ is semi-hyponormal and $\tilde{A}$ is hyponormal. Some authors paid attention to the relations between the spectral structure of A and $\hat{A}$ (e.g. [2], [3], [4]). In this note, we prove that for general $A \in p\text{-}\mathbf{H}, 0 < p \leq 1,\ A, \hat{A}$ and $\tilde{A}$ have "equal spectral structure", i.e. $\sigma_s(A) = \sigma_s(\hat{A}) = \sigma_s(\tilde{A})$, where $\sigma_s = \sigma, \sigma_a, \sigma_r, \sigma_{_B}, \sigma_w, \sigma_e, \sigma_k, \sigma_{_D}, \psi_n, \psi_{mn}$ or σ_p^0.

A subdecomposable operator is,up to similarity, the restriction of a decomposable operator to its invariant space. J. Eschmeier [5] proved that $A \in L(\mathbf{H})$ is subdecomposable if and only if $A \in (\beta)$, i.e. A has Bishop's property (β). M.Putinar and J.Eschmeier [6], [7] proved that hyponormal operators are subscalar and therefore subdecomposable. B.Duggal [2] asked whether a general p-hyponormal operator satisfies condition (β). We give an affirmative answer to this question. Yang Liming [8] proved that two quasisimilar hyponormal operators have equal essential spectra. By means of the subdecomposability of $A \in p\text{-}\mathbf{H}$, we generalize and strengthen this result to general p-hyponormal operators $(0 < p \leq 1)$.

For $T \in L(\mathbf{H}), \sigma(T), \sigma_p(T), \sigma_a(T)$, and $\sigma_e(T)$ denote the spectrum, point spectrum, approximate point spectrum and essential spectrum of T, respectively. Write

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$\nu(T) = \dim \mathrm{Ker} T$, $\mu(T) = \dim \mathrm{Ker} T^*$; the index of T is defined by $\mathrm{ind} T = \nu(T) - \mu(T)$ if at least one of $\nu(T)$ and $\mu(T)$ is finite. Let ψ denote the set of all semi-Fredholm operators on $\mathbf{H}$. Write $\rho_{s-F}(T) = \{\lambda \in \mathbf{C} : T - \lambda \in \psi\}$, $\psi_n(T) = \{\lambda \in \mathbf{C} : R(T-\lambda)$ is closed, ind $(T-\lambda) = n\}$ $(n = 0, \pm 1, \pm 2, \cdots, \pm\infty)$, $\psi_{mn}(T) = \{\lambda \in \mathbf{C} : R(T-\lambda)$ is closed, $\nu(T-\lambda) = m,\ \mu(T-\lambda) = n\}$ $(m, n = 0, 1, 2, \cdots, \infty)$. Also write $\psi_+(T) = \bigcup_{1 \le n \le \infty} \psi_n(T)$, $\psi_-(T) = \bigcup_{1 \le n \le \infty} \psi_{-n}(T)$, $\sigma_r(T) = \{\lambda \in \mathbf{C} :\ \ker(T-\lambda) = \{0\}, R(T-\lambda)$ is closed, $R(T-\lambda) \neq \mathbf{H}\}$, $\sigma_D(T) = \{\lambda \in \mathbf{C} : R(T-\lambda)$ is not closed$\}$, $\sigma_k(T) = \mathbf{C} \setminus \rho_{s-F}(T)$. $\sigma_p^0(T)$ denotes the set of all isolated eigenvalues of T with finite algebraic multiplicity. $\mathbf{K}(\mathbf{H})$ denotes the set of all compact operators on $\mathbf{H}$. $\sigma_B(T) = \bigcap_{\substack{K \in \mathbf{K}(\mathbf{H}) \\ KT=TK}} \sigma(T+K) = \sigma(T) \setminus \sigma_p^0(T)$, $\sigma_w(T) = \bigcap_{K \in \mathbf{K}(\mathbf{H})} \sigma(T+K) = \sigma(T) \setminus \psi_0(T)$.

Suppose $\lambda \in \mathbf{C}$, $T \in L(\mathbf{H})$. λ is called a regular point of the operator T if $\|P_{\mathrm{Ker}(T-\mu)} - P_{\mathrm{Ker}(T-\lambda)}\| \to 0$ $(\mu \to \lambda)$, where $P_{\mathrm{Ker}(T-\lambda)}$ denotes the orthogonal projection onto $\mathrm{Ker}(T-\lambda)$, $\tau^r(T)$ denotes the set of all regular points of T, $\tau^s(T) = \mathbf{C} \setminus \tau^r(T)$. For every set-valued function $B(\cdot) : L(\mathbf{H}) \to 2^{\mathbf{C}}$, write $B^r(T) = B(T) \cap \tau^r(T)$, $B^s(T) = B(T) \cap \tau^s(T)$ (see [9]).

Let $T \in L(\mathbf{H})$. Suppose that the closed subspace M of $\mathbf{H}$ reduces T; then M is said to be a normal subspace of T if $T|_M$ is a normal operator. The operator T is said to be pure if T has no non-trival normal subspace.

$\mathcal{O}(\Omega, \mathbf{H})$ denotes the Fréchet space of all $\mathbf{H}$-valued analytic functions on the open set $\Omega \subset \mathbf{C}$ with the topology defined by uniform convergence on every compact subset of Ω. Suppose $T \in L(\mathbf{H})$; T is said to have Bishop's property (β) (denoted by $T \in (\beta)$) if the mapping $\alpha_{T,\Omega} : \mathcal{O}(\Omega, \mathbf{H}) \to \mathcal{O}(\Omega, \mathbf{H}), f \mapsto (T-z)f$ is injective and has closed range for every open subset Ω of $\mathbf{C}$. Write $E_2(T) = \{\lambda \in \mathbf{C} : \exists \delta > 0$ such that for $\Omega = O(\lambda, \delta'), 0 < \delta' < \delta$, $\alpha_{T,\Omega}$ has closed range$\}$, $A(T) = \{\lambda \in \mathbf{C} :\ \exists \delta > 0$ such that for $\Omega = O(\lambda, \delta'), 0 < \delta' < \delta$, $\alpha_{T,\Omega}$ is injective$\}$. Write $T \in (E_2)$ $((A))$ if for every $\lambda \in \mathbf{C}, \lambda \in E_2(T)$ $(A(T))$. $T \in (A)$ is equivalent to T has the single-valued extension property. It is clear by definitions (see [10, Proposition 4]) that $T \in (\beta)$ if and only if $T \in (A)$ and $T \in (E_2)$.

Let $T_1, T_2 \in L(\mathbf{H})$; we say T_1 is a dense (quasiaffine) transform of T_2, denoted as $T_1 \xrightarrow{dr} T_2$ $(T_1 \xrightarrow{q} T_2)$, if there exists operator X with dense range (injective and dense range) such that $XT_1 = T_2X$. We said T_1 and T_2 are densely (quasi-) similar, denoted as $T_1 \overset{dr}{\sim} T_2$ $(T_1 \overset{q}{\sim} T_2)$, if $T_1 \xrightarrow{dr} T_2 \xrightarrow{dr} T_1$ $(T_1 \xrightarrow{q} T_2 \xrightarrow{q} T_1)$.

2. Spectral structure of A and $\hat{A}$

Lemma 1. *If T is a pure p-hyponormal operator, then $\sigma_p(T) = \emptyset$.*

Proof. Let T have the polar decomposition $T = U|T|$. If $\lambda \in \sigma_p(T)$, then $\mathrm{Ker}(T-\lambda) \neq \{0\}$. By [11, Proof of Theorem 4], $\mathrm{Ker}(T-\lambda) \subset \mathrm{Ker}(T-\lambda)^*$. This implies that $\mathrm{Ker}(T-\lambda)$ is a reducing subspace of T and $T|_{\mathrm{Ker}(T-\lambda)}$ is normal, a contradiction to the purity of T. Hence $\sigma_p(T) = \emptyset$. □

Lemma 2. *Let $S, T \in L(\mathbf{H})$. If $A = TS, B = ST$, then*

$$\dim \mathrm{Ker}(A-\lambda) = \dim \mathrm{Ker}(B-\lambda), \qquad \lambda \neq 0;$$

moreover, if $\mathrm{Ker}\, S = \mathrm{Ker}\, T$*, then* $\sigma_p(A) = \sigma_p(B)$.

Proof. Suppose $\lambda \neq 0$, and $x_i \in \mathrm{Ker}(A-\lambda), i = 1, 2, \cdots, n$; $x_1, x_2, \cdots, x_n$ are linearly independent. Then $BSx_i = STSx_i = SAx_i = \lambda Sx_i$, $Sx_i \in$

$\mathrm{Ker}(B-\lambda)$, $i=1,2,\cdots,n$. We claim that $Sx_1, Sx_2, \cdots, Sx_n$ are linearly independent too. For otherwise, there would exist constants $\alpha_1, \alpha_2, \cdots, \alpha_n$, not all zero, such that $\sum_{i=1}^n \alpha_i Sx_i = 0$ and hence $\sum_{i=1}^n \alpha_i Ax_i = \sum_{i=1}^n \alpha_i TSx_i = 0$. Since $\sum_{i=1}^n \alpha_i Ax_i = \lambda \sum_{i=1}^n \alpha_i x_i$, and $\lambda \neq 0$, therefore $\sum_{i=1}^n \alpha_i x_i = 0$, a contradiction with the supposition that $x_1, x_2, \cdots, x_n$ are linearly independent. Therefore $\dim\mathrm{Ker}(A-\lambda) \leq \dim\mathrm{Ker}(B-\lambda)$. A similar argument shows that $\dim\mathrm{Ker}(B-\lambda) \leq \dim\mathrm{Ker}(A-\lambda)$. It follows that $\dim\mathrm{Ker}(A-\lambda) = \dim\mathrm{Ker}(B-\lambda)$, $\lambda \neq 0$.

This equality implies $\sigma_p(A)\setminus\{0\} = \sigma_p(B)\setminus\{0\}$.

If $0 \in \sigma_p(A)$, then there exists $x \in \mathbf{H}, x \neq 0$ and $Ax = TSx = 0$. This implies $BSx = 0$. If $Sx \neq 0$, then $0 \in \sigma_p(B)$. If $Sx = 0$, then it follows from $\mathrm{Ker}\, S = \mathrm{Ker}\, T$ that $Tx = 0$ and $Bx = 0$; this implies $0 \in \sigma_p(B)$, too. Similarly $0 \in \sigma_p(B)$ implies $0 \in \sigma_p(A)$. The conclusion $\sigma_p(A) = \sigma_p(B)$ now follows. □

Lemma 3. *Suppose that $A \in L(\mathbf{H})$; then* $\dim\mathrm{Ker}(A-\lambda) = \dim\mathrm{Ker}(\hat{A}-\lambda)$, $\dim\mathrm{Ker}(A-\lambda)^* = \dim\mathrm{Ker}(\hat{A}-\lambda)^*$, $\lambda \neq 0$, *and* $\sigma_p(A) = \sigma_p(\hat{A})$.

Proof. Since $A = U|A| = U|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}, \hat{A} = |A|^{\frac{1}{2}} \cdot U|A|^{\frac{1}{2}}, A^* = |A|^{\frac{1}{2}} \cdot |A|^{\frac{1}{2}}U^*, (\hat{A})^* = |A|^{\frac{1}{2}}U^* \cdot |A|^{\frac{1}{2}}$ and $\mathrm{Ker}|A|^{\frac{1}{2}} = \mathrm{Ker}(U|A|^{\frac{1}{2}})$, the conclusions follow from Lemma 2. □

Lemma 4 ([12])**.** *Let $T \in L(\mathbf{H})$ be a semi-Fredholm operator. Then there exists $\delta > 0$ such that $S \in L(\mathbf{H}), \|T-S\| < \delta$ implies that S is semi-Fredholm and $\nu(S) \leq \nu(T)$, $\mu(S) \leq \mu(T), \mathrm{ind}\, S = \mathrm{ind}\, T$.*

Theorem 5. *Let $A \in L(\mathbf{H})$. If A_0, the pure part of A, has no eigenvalue, then*

(i) $\sigma(A) = \sigma_p^0(A) \cup (\bigcup_{1\leq n\leq\infty} \psi_{0n}(A)) \cup \psi_-^s(A) \cup \psi_{\infty\infty}^s(A) \cup \sigma_D(A)$,

(ii) A and $\hat{A}$ have "equal spectral structure", i.e. $\sigma_s(A) = \sigma_s(\hat{A})$, where $\sigma_s = \sigma, \sigma_a, \sigma_r, \sigma_B, \sigma_w, \sigma_e, \sigma_k, \sigma_D, \psi_n$ $(-\infty \leq n \leq \infty), \psi_{mn}$ $(0 \leq m, n \leq \infty)$ or σ_p^0.

Proof. Decompose A into normal and pure parts: $A = N \oplus A_0$. If $N = W|N|$ and $A_0 = V|A_0|$ are the polar decompositions of N and A_0 respectively, then it is easy to derive that $W|N| = |N|W, \hat{N} = |N|^{\frac{1}{2}}W|N|^{\frac{1}{2}} = N$ and that $\hat{A} = N \oplus \hat{A}_0$.

Since $\sigma_p(\hat{A}_0) = \sigma_p(A_0) = \emptyset$,we have

$$\sigma(A_0) = (\bigcup_{1\leq n\leq\infty} \psi_{0n}(A_0)) \cup \sigma_D(A_0), \tag{1}$$

$$\sigma(\hat{A}_0) = (\bigcup_{1\leq n\leq\infty} \psi_{0n}(\hat{A}_0)) \cup \sigma_D(\hat{A}_0). \tag{2}$$

By the property of normal operators, we have

$$\sigma(N) = \sigma_p^0(N) \cup \psi_{\infty\infty}^s(N) \cup \sigma_D(N). \tag{3}$$

(1) and (3) implies that (see [9])

$$\sigma(A) = \sigma_p^0(A) \cup (\bigcup_{1\leq n\leq\infty} \psi_{0n}(A)) \cup \psi_-^s(A) \cup \psi_{\infty\infty}^s(A) \cup \sigma_D(A),$$

i.e. (i) holds.

Now let us turn to the spectral parts of A_0 and $\hat{A}_0$. Let A_0^o, $(\hat{A}_0)^o = \hat{A}_0^o$ be the Berberian extension (see [13], Chapter I) of A_0 and $\hat{A}_0$ respectively; then

$\sigma_a(A_0) = \sigma_p(A_0^o)$ and $\sigma_a(\hat{A}_0) = \sigma_p(\hat{A}_0^o)$. By Lemma 3, $\sigma_p(A_0^o) = \sigma_p(\hat{A}_0^o)$; by $(1),(2), \sigma_a(A_0) = \sigma_D(A_0)$ and $\sigma_a(\hat{A}_0) = \sigma_D(\hat{A}_0)$. It follows then that

$$\sigma_D(A_0) = \sigma_D(\hat{A}_0). \tag{4}$$

It follows from Lemma 3 and (4) that

$$\psi_{0n}(A_0) \setminus \{0\} = \psi_{0n}(\hat{A}_0) \setminus \{0\},\ 0 \le n \le \infty.$$

Since $\psi_{0n}(A_0)$ and $\psi_{0n}(\hat{A}_0)$ are open sets by Lemma 4, and $\sigma_D(A_0) = \sigma_D(\hat{A}_0)$, it is easy to derive that

$$\psi_{0n}(A_0) = \psi_{0n}(\hat{A}_0),\ 0 \le n \le \infty. \tag{5}$$

$(1) - (5)$ imply that

$$\sigma_D(A) = \sigma_D(\hat{A}), \tag{6}$$

$$\psi_{mn}(A) = \psi_{mn}(\hat{A}) = \emptyset,\ m > n,$$

$$\psi_{mn}(A) = \psi_{mm}(N) \cap \psi_{0,n-m}(A_0) = \psi_{mm}(N) \cap \psi_{0,n-m}(\hat{A}_0) = \psi_{mn}(\hat{A}),\quad 0 \le m \le n, m < \infty.$$

$$\psi_{\infty\infty}(A) = \psi^s_{\infty\infty}(N) \setminus \sigma_D(A_0) = \psi^s_{\infty\infty}(N) \setminus \sigma_D(\hat{A}_0) = \psi_{\infty\infty}(\hat{A}),$$

or briefly,

$$\psi_{mn}(A) = \psi_{mn}(\hat{A}),\ 0 \le m, n \le \infty. \tag{7}$$

The equalities $\sigma_s(A) = \sigma_s(\hat{A})$ in (ii) come now immediately from $(6), (7)$ and the definitions and fundamental properties of various σ_s. ☐

Theorem 6. *If $A \in p$-$\mathbf{H}$, $0 < p \le 1$, then A has "equal spectral structure" (see Theorem 5) with the semi-hyponormal operator $\hat{A}$ and the hyponormal operator $\tilde{A}$.*

Proof. Obvious from Lemma 1 and Theorem 5. ☐

3. Subdecomposability and quasisimilarity

Theorem 7. *Let $T \in L(\mathbf{H}), \lambda \in \mathbf{C}$. If $\operatorname{Ker} T \subset \operatorname{Ker} T^*$, then*
(1) $\lambda \in A(T)$ if and only if $\lambda \in A(\hat{T})$,
(2) $\lambda \in E_2(T)$ if and only if $\lambda \in E_2(\hat{T})$,
(3) $T \in (\beta)$ if and only if $\hat{T} \in (\beta)$.

Proof. Since $\mathrm{Ker}T \subset \mathrm{Ker}T^*$, we can write $T = \theta \oplus T_1$, where $\theta = T|_{\mathrm{ker}T}, T_1 = T|_{(\mathrm{Ker}T)^\perp}$. Write $\mathbf{H}_1 = (\mathrm{Ker}T)^\perp$ and let $T_1 = U|T_1|$ be the polar decomposition. It is clear that $\mathrm{Ker}T_1 = \{0\}$ and $\hat{T} = \theta \oplus \hat{T}_1$. Thus, to prove the required result it suffices to show that $\lambda \in A(T_1)$ $(E_2(T_1))$ if and only if $\lambda \in A(\hat{T}_1)$ $(E_2(\hat{T}_1))$.

(1) Suppose that $\lambda \in A(T_1)$; then there exists $\delta > 0$ such that for $\Omega = O(\lambda, \delta')$, $0 < \delta' < \delta, \alpha_{T_1,\Omega}$ is injective. Let $f \in \mathcal{O}(\Omega, \mathbf{H}_1)$, $(\hat{T}_1 - z)f(z) = 0 (z \in \Omega)$. Then $(T_1 - z)U|T_1|^{\frac{1}{2}} f(z) = U|T_1|^{\frac{1}{2}}(\hat{T}_1 - z)f(z) = 0$. Since $\alpha_{T_1,\Omega}$ is injective, $U|T_1|^{\frac{1}{2}} f(z) = 0$ $(z \in \Omega)$. It follows from $\mathrm{Ker}U|T_1|^{\frac{1}{2}} = \mathrm{Ker}T_1 = \{0\}$ that $f(z) = 0$ $(z \in \Omega)$. Thus $\lambda \in A(\hat{T}_1)$.

The argument for the converse statement is similar.

(2) Suppose that $\lambda \in E_2(\hat{T}_1)$; then there exists $\delta > 0$ such that for $\Omega = O(\lambda, \delta'), 0 < \delta' < \delta, \alpha_{\hat{T}_1,\Omega}$ has closed range. Suppose that $f_n \in \mathcal{O}(\Omega, \mathbf{H}_1), n = 1, 2, ..., (T_1 - z)f_n \to g \in \mathcal{O}(\Omega, \mathbf{H}_1)$; then $|T_1|^{\frac{1}{2}}(U|T_1| - z)f_n \to |T_1|^{\frac{1}{2}}g$, i.e. $(\hat{T}_1 - z)|T_1|^{\frac{1}{2}}f_n \to |T_1|^{\frac{1}{2}}g$. By the hypothesis $\lambda \in E_2(\hat{T}_1)$, there exists $f \in \mathcal{O}(\Omega, \mathbf{H}_1)$ such that $|T_1|^{\frac{1}{2}}g = (\hat{T}_1 - z)f$. A simple calculation shows that

$$f(z) = \frac{1}{z}(\hat{T}_1 f(z) - |T_1|^{\frac{1}{2}}g(z)) = |T_1|^{\frac{1}{2}}(\frac{U|T_1|^{\frac{1}{2}}f(z) - g(z)}{z}) \qquad (z \in \Omega \setminus \{0\}).$$

If $\lambda \neq 0$, we may assume that $0 \notin \Omega$. Let $\phi(z) = \frac{U|T_1|^{\frac{1}{2}}f(z) - g(z)}{z}$ $(z \in \Omega)$. It is obvious then that $\phi \in \mathcal{O}(\Omega, \mathbf{H}_1)$ and $|T_1|^{\frac{1}{2}}g = (\hat{T}_1 - z)|T_1|^{\frac{1}{2}}\phi = |T_1|^{\frac{1}{2}}(T_1 - z)\phi$. But since $\mathrm{Ker}|T_1|^{\frac{1}{2}} = \mathrm{Ker}T_1 = \{0\}$, hence $g = (T_1 - z)\phi$, and so $\alpha_{T_1,\Omega}$ has closed range. Thus $\lambda \in E_2(T_1)$.

If $\lambda = 0$, let $h(z) = U|T_1|^{\frac{1}{2}}f(z) - g(z)$ $(z \in \Omega)$; then $h \in \mathcal{O}(\Omega, \mathbf{H}_1)$. Since $|T_1|^{\frac{1}{2}}g(0) = |T_1|^{\frac{1}{2}}U|T_1|^{\frac{1}{2}}f(0)$ and $\mathrm{Ker}|T_1|^{\frac{1}{2}} = \mathrm{Ker}T_1 = \{0\}$, we have

$$h(0) = U|T_1|^{\frac{1}{2}}f(0) - g(0) = 0.$$

Define

$$\phi(z) = \begin{cases} h(z)/z\ , & 0 \neq z \in \Omega, \\ h'(0), & z = 0. \end{cases}$$

It can be verified by calculation that $\phi \in \mathcal{O}(\Omega, \mathbf{H}_1)$. It follows from the preceding section that $g(z) = (T_1 - z)\phi(z)$ $(z \in \Omega \setminus \{0\})$ and this, by virtue of the analyticity of $g(z), \phi(z)$, implies that $g(z) = (T_1 - z)\phi(z)$ $(z \in \Omega)$, and so $\alpha_{T_1,\Omega}$ has closed range. Thus $0 \in E_2(T_1)$.

The converse argument is similar.

(3) The statement is obvious from (1),(2) and $T \in (\beta)$ if and only if $T \in (A)$ and $T \in (E_2)$. □

Theorem 8. *Suppose that $A \in p$-$\mathbf{H}, 0 < p \leq 1$; then $A \in (\beta)$, i.e. A is subdecomposable.*

Proof. Suppose $A \in p$-$\mathbf{H}$; then $\mathrm{Ker}A \subset \mathrm{Ker}A^*$ and $\mathrm{Ker}\hat{A} \subset \mathrm{Ker}(\hat{A})^*$ by [11, Lemma 1]. Being a hyponormal operator, $\tilde{A} \in (\beta)$; consequently $\hat{A} \in (\beta)$ and hence $A \in (\beta)$ by Theorem 7,(3). □

Lemma 9 ([10, Corollary 3]). *Let $S \in L(\mathbf{H})$ be subdecomposable, $T \in L(\mathbf{H})$. If $T \xrightarrow{dr} S$, then $\sigma(S) \subset \sigma(T)$; if $T \overset{dr}{\sim} S$, then $\sigma_e(S) \subset \sigma_e(T)$.*

Theorem 10. *Let $A \in p$-$\mathbf{H}, 0 < p \leq 1, B \in L(\mathbf{H})$. If $B \xrightarrow{dr} A$, then $\sigma(A) \subset \sigma(B)$; if $B \overset{dr}{\sim} A$, then $\sigma_e(A) \subset \sigma_e(B)$.*

Proof. Obvious from Theorem 8 and Lemma 9. □

Corollary 11. *If A or $A^* \in p$-$\mathbf{H}, 0 < p \leq 1, B \in L(\mathbf{H}), A \overset{q}{\sim} B$, then $\sigma_e(A) \subset \sigma_e(B)$, $\sigma(A) \subset \sigma(B)$.*

Corollary 12. *If $A, B \in p$-$\mathbf{H}, 0 < p \leq 1, A \overset{q}{\sim} B$, then $\sigma_e(A) = \sigma_e(B)$, $\sigma(A) = \sigma(B)$.*

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Department of Mathematics, Fujian Normal University, Fuzhou, 350007, The People's Republic of China

E-mail address: xhyan@fjtu.edu.cn