

ON ORDER CONTINUOUS NORMS

SI-KIT CHUNG AND DENNY LEUNG

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ABSTRACT. It is shown that a normed vector lattice $(E, \|\cdot\|)$ is order continuous if and only if, for every lattice norm ρ on E with $\rho \leq \|\cdot\|$, the $\|\cdot\|$ -topology and ρ -topology coincide on every order interval of E .

1. INTRODUCTION AND PRELIMINARIES

A normed vector lattice E is said to be order continuous, respectively, σ -order continuous, if $\lim_{\alpha} \|x_{\alpha}\| = 0$ for every downward directed system, respectively, decreasing sequence, $(x_{\alpha})_{\alpha \in \Lambda}$ in E such that $\bigwedge_{\alpha \in \Lambda} x_{\alpha} = 0$. It is said to satisfy the norm-Cauchy condition if every order bounded increasing sequence in E is Cauchy. It is well-known that a normed vector lattice is order continuous if and only if it is σ -order continuous and satisfies the norm-Cauchy condition (see, e.g., [6, Theorem 103.4]). It is easy to observe that both σ -order continuity of the norm and the norm-Cauchy condition pass from a normed vector lattice to its norm completion. Hence, the norm completion of an order continuous normed vector lattice is also order continuous. Banach lattices, or more generally, normed vector lattices, with order continuous norms have been studied extensively and various characterizations of order continuity have been obtained (see [1, 5, 6]). The starting point of the present paper is the following result due to Amemiya (cf. [4, Theorem 2.4.8]).

Theorem 1.1. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two order continuous norms on a vector lattice E , and suppose that $(E, \|\cdot\|_i)$ is an ideal in its norm completion for $i = 1, 2$. Then on every order interval of E , the norm topologies coincide and the weak topologies coincide.*

We will show that the theorem holds without the assumption that $(E, \|\cdot\|_i)$ is an ideal in its norm completion. In fact, the conclusion of Amemiya's Theorem gives a characterization of the order continuity of the norms involved (see Theorem 2.3). We begin with some examples which show that the conclusion of Amemiya's Theorem no longer holds under certain weakenings of the hypothesis.

Example 1.1. Let E be the vector lattice $C[0, 1]$. Let $\|\cdot\|_1$ be the L^1 -norm, and let $\|\cdot\|_2$ be given by $\|f\|_2 = \int_A |f|$, where A is a measurable subset of $[0, 1]$ with measure less than 1, but the measure of the intersection of A with any open interval contained in $[0, 1]$ is positive. Clearly, both $\|\cdot\|_1$ and $\|\cdot\|_2$ are lattice norms on

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E which satisfy the norm-Cauchy condition. However, the topologies generated by the two norms differ on the order interval $[0, \mathbf{1}]$, where $\mathbf{1}$ denotes the constant 1 function.

Example 1.2. Let E be the vector lattice of all functions in $L^1[0, 1]$ which is (equal almost everywhere to a function that is) continuous at 0. If $f \in E$, let $\rho(f) = |g(0)|$, where $g = f$ almost everywhere, and g is continuous at 0. It is clear that ρ is a well-defined lattice seminorm on E . Let $\|\cdot\|_1$ be the L^1 -norm, and let $\|\cdot\|_2 = \|\cdot\|_1 + \rho$. Then $\|\cdot\|_1$ is an order continuous norm on E , and $\|\cdot\|_2$ is a lattice norm on E which satisfies the norm-Cauchy condition. However, the topologies generated by $\|\cdot\|_1$ and $\|\cdot\|_2$ differ on $[0, \mathbf{1}]$, where $\mathbf{1}$ is the constant 1 function.

Example 1.3. Let E be a vector lattice, and suppose $\|\cdot\|_1$ is a σ -order continuous norm on E which is not order continuous. By Theorem 2.3, there exists a lattice norm $\|\cdot\|_2$ on E such that $\|\cdot\|_2 \leq \|\cdot\|_1$, and the topologies generated by the two norms differ on some order interval of E . Thus, the conclusion of Amemiya's Theorem may still fail to hold even if both norms are assumed to be σ -order continuous.

2. MAIN RESULTS

In this section, we give the proof of the generalization of Amemiya's Theorem mentioned above, as well as certain characterizations of order continuity in normed vector lattices. The way in which order continuity is applied in this circle of ideas is contained in the next proposition.

Proposition 2.1. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two lattice norms on a vector lattice E . For $\alpha = 1, 2$, let \widehat{E}_α be the norm completion of $E_\alpha = (E, \|\cdot\|_\alpha)$. Suppose the identity map $i : E_1 \rightarrow E_2$ is continuous and E_1 is order continuous. Then the extension $\hat{i} : \widehat{E}_1 \rightarrow \widehat{E}_2$ of i is one-to-one.*

Proof. Let F be the $\sigma(E'_1, \widehat{E}_1)$ -closure of $i'(E'_2)$. Then F is a band in E'_1 . Since E_1 is order continuous, by [6, Corollary 106.4], F is $\sigma(E'_1, E_1)$ -closed. As i is one-to-one, $i'(E'_2)$ is $\sigma(E'_1, E_1)$ -dense in E'_1 . Therefore, $F = E'_1$. Hence $i'(E'_2)$ is $\sigma(E'_1, \widehat{E}_1)$ -dense in E'_1 . It follows that \hat{i} is one-to-one. \square

We are now ready to prove the promised generalization of Amemiya's Theorem.

Theorem 2.2. *Let E be a vector lattice. Suppose $\|\cdot\|_1$, respectively $\|\cdot\|_2$, is an order continuous norm, respectively, σ -order continuous norm, on E . Then on every order interval of E , the $\|\cdot\|_1$ -topology and $\|\cdot\|_2$ -topology coincide and the $\sigma(E, (E, \|\cdot\|_1)')$ -topology and $\sigma(E, (E, \|\cdot\|_2)')$ -topology coincide.*

Proof. First, we show that $\|\cdot\|_2$ is in fact order continuous. Let $(x_\alpha)_{\alpha \in \Lambda}$ be a decreasing net in E with $\inf_{\alpha \in \Lambda} x_\alpha = 0$. Fix $\alpha_0 \in \Lambda$. Let E_{α_0} be the principal ideal of E generated by x_{α_0} . Since $(E, \|\cdot\|_1)$ is order continuous, by [6, Theorem 103.12], there exists $x' \in (E, \|\cdot\|_1)'_+$ such that x' is strictly positive on E_{α_0} . Since $\lim_{\alpha \in \Lambda} \|x_\alpha\|_1 = 0$, $\lim_{\alpha \in \Lambda} \langle x_\alpha, x' \rangle = 0$. Hence there exist $\alpha_n \in \Lambda, n \in \mathbf{N}$, with $\alpha_{n+1} \leq \alpha_n$ for all $n \in \mathbf{N}$ such that $\lim_{n \rightarrow \infty} \langle x_{\alpha_n}, x' \rangle = 0$. Since x' is strictly positive on E_{α_0} , we have $\inf_{n \in \mathbf{N}} x_{\alpha_n} = 0$. The σ -order continuity of $\|\cdot\|_2$ implies that $\inf_{n \in \mathbf{N}} \|x_{\alpha_n}\|_2 = 0$ and so $\inf_{\alpha \in \Lambda} \|x_\alpha\|_2 = 0$.

Next we prove the required result for the case where $(E, \|\cdot\|_1)$ is a Banach lattice and $\|\cdot\|_1 \geq \|\cdot\|_2$. Denote the weak topologies $\sigma(E, (E, \|\cdot\|_1)')$ and $\sigma(E, (E, \|\cdot\|_2)')$ by σ_1 and σ_2 respectively. Let $x \in E_+$. Since $(E, \|\cdot\|_1)$ is an order continuous

Banach lattice, the order interval $[0, x]$ is σ_1 -compact. Note that the identity map $i : ([0, x], \sigma_1) \rightarrow ([0, x], \sigma_2)$ is continuous and that σ_2 is Hausdorff, so $\sigma_1 = \sigma_2$ on $[0, x]$. Suppose (x_n) is a sequence in $[0, x]$ with $\lim_{n \rightarrow \infty} \|x_n\|_2 = 0$. Then $x_n \xrightarrow{\sigma_2} 0$ and so $x_n \xrightarrow{\sigma_1} 0$. By [4, Proposition 2.3.4], there exists a disjoint sequence (x'_n) in the positive part of the unit ball of $(E, \|\cdot\|_1)'$ such that $\limsup_{n \rightarrow \infty} \langle x_n, x'_n \rangle = \limsup_{n \rightarrow \infty} \|x_n\|_1$. By [4, Corollary 2.4.3], we have $\lim_{n \rightarrow \infty} \langle x, x'_n \rangle = 0$. Therefore $\lim_{n \rightarrow \infty} \langle x_n, x'_n \rangle = 0$ and so $\lim_{n \rightarrow \infty} \|x_n\|_1 = 0$.

For the general case, we let $\rho = \|\cdot\|_1 + \|\cdot\|_2$. It suffices to prove the required result for ρ and $\|\cdot\|_1$. Let E_0 and E_1 be the norm completions of (E, ρ) and $(E, \|\cdot\|_1)$ respectively (with the norms still denoted by ρ and $\|\cdot\|_1$ respectively). By Proposition 2.1, the extension $\hat{i} : (E_0, \rho) \rightarrow (E_1, \|\cdot\|_1)$ of the identity map $i : (E, \rho) \rightarrow (E, \|\cdot\|_1)$ is one-to-one. We may identify E_0 and $\hat{i}(E_0)$ and so we have two lattice norms ρ and $\|\cdot\|_1$ on E_0 . Since E_0 is the norm completion of the order continuous normed vector lattice (E, ρ) , (E_0, ρ) is order continuous. The required result then follows from the result on the special case above. □

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a normed vector lattice, and let \widehat{E} be its norm completion. The following assertions are equivalent.*

- (1) $(E, \|\cdot\|)$ is order continuous.
- (2) For every lattice norm ρ on E with $\rho \leq \|\cdot\|$, the $\|\cdot\|$ -topology and ρ -topology coincide on every order interval of E .
- (3) E is strongly order dense in \widehat{E} , and \widehat{E} is order continuous.

Proof. (3) \implies (1) Obvious.

(1) \implies (2) This follows easily from Theorem 2.2.

(2) \implies (3) We divide the proof into two parts.

(I) Suppose $(\widehat{E}, \|\cdot\|)$ is not order continuous. Then by [3, Satz 8], there exists a disjoint sequence (x_n) in \widehat{E} contained in an order interval $[0, y]$ of \widehat{E} such that $\|x_n\| = 1$ for all $n \in \mathbf{N}$. We may assume that $y \in E$. Let P_n (respectively P) be the band projection from \widehat{E}'' onto the band generated in \widehat{E}'' by x_n (respectively $\{x_n : n \in \mathbf{N}\}$). Define a lattice norm ρ on \widehat{E} by $\rho(x) = \frac{1}{2}(\|x - Px\| + \sum_{n=1}^{\infty} \frac{\|P_n x\|}{2^n})$. Then $\rho \leq \|\cdot\|$ on \widehat{E} and $\rho(x_n) = \frac{1}{2^{n+1}}$. Choose a sequence (y_n) in E such that $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$. We may assume $0 \leq y_n \leq y$. Since $\rho(y_n) \leq \rho(y_n - x_n) + \rho(x_n) \leq \|y_n - x_n\| + \rho(x_n) \rightarrow 0$ as $n \rightarrow \infty$, it follows from the assumption on E that $\lim_{n \rightarrow \infty} \|y_n\| = 0$. Hence we have $\lim_{n \rightarrow \infty} \|x_n\| = 0$, a contradiction.

(II) Suppose E is not strongly order dense in \widehat{E} . Let $0 < x \in \widehat{E}$ be such that $[0, x] \cap E = \{0\}$. We may assume that $x < y$ for some $y \in E$. Let B be the band in \widehat{E} generated by y and B^\perp be its disjoint complement. Since $(\widehat{E}, \|\cdot\|)$ is order continuous, by [2, Theorem 1.b.14], B may be represented as an ideal of $L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ) so that y corresponds to the constant 1 function and B' consists of all measurable functions g such that $\int_\Omega |gh| d\mu < \infty$ for all $h \in B$.

Fix $\epsilon > 0$. Let $A = \{\omega \in \Omega : x(\omega) < \epsilon\}$. Note that χ_A , that is, the map given by $h \mapsto \int_A h d\mu$ for $h \in B$, belongs to $(B, \|\cdot\|)'$. Define a lattice semi-norm ρ_A on \widehat{E} by $\rho_A(h + z) = \max\{(\|\chi_A\|_{(B, \|\cdot\|)'} + 1)^{-1} \int_A |h| d\mu, \|z\|\}$ for $h \in B, z \in B^\perp$. It is clear that $\rho_A \leq \|\cdot\|$ on \widehat{E} . Moreover, ρ_A is a lattice norm on E . Indeed, if $h + z \in E$ where $h \in B, z \in B^\perp$ and $\rho_A(h + z) = 0$, then $z = 0$ and $\int_A |h| d\mu = 0$. So $h = 0$ a.e. on A . Hence $\epsilon(|h| \wedge y) \in [0, x] \cap E$. Therefore $|h| \wedge y = 0$ and so $h = 0$.

Since $0 \leq \chi_{\Omega \setminus A} \in B$, a sequence (y_n) exists in E_+ such that

$$\lim_{n \rightarrow \infty} \|y_n - \chi_{\Omega \setminus A}\| = 0.$$

Let $h_n = y_n \wedge y$. Then $\lim_{n \rightarrow \infty} \|h_n - \chi_{\Omega \setminus A}\| = 0$. So $\rho_A(h_n) = \int_A h_n d\mu = \int_A (h_n - \chi_{\Omega \setminus A}) d\mu \leq \|\chi_A\|_{(B, \|\cdot\|)'} \|h_n - \chi_{\Omega \setminus A}\| \rightarrow 0$ as $n \rightarrow \infty$. Hence by the assumption on E , $\lim_{n \rightarrow \infty} \|h_n\| = 0$ and so $\chi_{\Omega \setminus A} = 0$. Thus

$$\mu(\{\omega \in \Omega : \epsilon \leq x(\omega)\}) = 0 \quad \text{for all } \epsilon > 0$$

and so we have $x = 0$, a contradiction. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG, HONG KONG
E-mail address: chung@submaths.hku.hk

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE 119260,
 SINGAPORE
E-mail address: mat1hh@nus.edu.sg