ON BURGESS’S THEOREM AND RELATED PROBLEMS

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Abstract. Let $G$ be a graph. We determine all graphs which are $G$-like. We also prove that if $G_i$ $(i = 1, 2, \ldots, m)$ are graphs, then in order that each $G_i$-like $(i = 1, 2, \ldots, m)$ continuum $M$ be $n$-indecomposable for some $n = n(M)$ it is necessary and sufficient that if $K$ is a graph, then $K$ is not $G_i$-like for some integer $i$ with $1 \leq i \leq m$. This generalizes a well known theorem of Burgess.

1. Introduction

In this paper we study the structures of graph-like graphs and the structures of a finitely-many-graphs-like continua. Namely, if $G$ is a graph, we determine all graphs which are $G$-like. We also prove that if $G_i$ $(i = 1, 2, \ldots, m)$ are graphs, then in order that each $G_i$-like $(i = 1, 2, \ldots, m)$ continuum $M$ be $n$-indecomposable for some $n = n(M)$ it is necessary and sufficient that if $K$ is a graph, then $K$ is not $G_i$-like for some integer $i$ with $1 \leq i \leq m$. This generalizes a well known result of Burgess. The results will be used in a forthcoming paper by the same authors in determining the set of periods of a piecewise monotone map of a graph (see [LXY] for some background).

By a continuum we mean a non-empty connected compact metric space. A continuum $M$ is decomposable (resp., indecomposable) if it is (resp., is not) the union of its two proper subcontinua. Let $X$, $Y$ be continua and $d$ be a metric on $X$. A continuous surjective map $f : X \rightarrow Y$ is an $\epsilon$-map if for each $y \in Y$, $\text{diam}(f^{-1}(y)) < \epsilon$. If for each $\epsilon > 0$ there is an $\epsilon$-map from $X$ onto $Y$, then we say $X$ is $Y$-like.

A continuum $M$ is said to be the essential sum of some collection of its subcontinua if the union of the collection is $M$ and there is no element of the collection such that it is contained in the union of the rest of the elements from the collection. If $n \in \mathbb{N}$ and the continuum $M$ is the essential sum of $n$ continua and not the essential sum of $n + 1$ continua, then $M$ is said to be $n$-indecomposable. It is known that for any such continuum $M$, there is a unique collection consisting of $n$ indecomposable continua having $M$ as their essential sum ([BI]).

By a graph we mean a connected compact one-dimensional branch manifold. Let $G$ be a graph. For $x \in G$, there is a closed connected neighbourhood $V$ of $x$ such
that if \( V' \) is a closed connected neighbourhood of \( x \) contained in \( V \), then \( V' \) is homeomorphic to \( V \). 

**Proof.** Let \( G \) be a graph. If \( V \) is a graph-like, then the continuum is a graph, and hence generalize some result of [MS]. We start with the following definition.

**Definition 2.1.** Let \( G, K \) be graphs. We say that \( K \approx G \) if there are pairwise disjoint subgraphs of \( G \) such that \( K \) is homeomorphic to the graph obtained by shrinking the subgraphs to points.

An immediate observation is

**Remark 2.2.** Let \( G, K \) be graphs. If \( K \approx G \), then \( E(K) + B(K) \leq E(G) + B(G) \).

With the above definition we now show the main result of the section.

**Theorem 2.3.** Let \( G \) be a graph. Then a graph \( K \) is graph-like if and only if \( K \approx G \).

**Proof.** Let \( d \) be a metric on \( K \). First we show that if \( K \approx G \), then \( K \) is graph-like. Assume that \( K \) is the graph obtained by shrinking subgraphs \( G_1, \ldots, G_n \) (of \( G \)) to points.

Let \( q : G \to K \) be the quotient map and \( q(G_i) = \{ x_i \} \), \( 1 \leq i \leq n \). Take a connected closed small neighbourhood \( U_i \) of \( x_i \) which is homeomorphic to \( n_i \)-star, where \( n_i = Val(x_i) \). Furthermore, take a connected closed small neighbourhood \( V_i \) of \( G_i \) which has \( n_i \)-end points. Let \( \varepsilon > 0 \). Then an \( \varepsilon \)-map \( g \) from \( K \) onto \( G \) can be obtained by taking the union of \( g|_{K \setminus U_i} = q^{-1}_{\mid G_i} V_i \), with an easily constructed \( \varepsilon \)-map from \( U_i \) onto \( V_i \), \( 1 \leq i \leq n \).

Now we prove that if \( K \) is graph-like, then \( K \approx G \).

Let \( n \) be the number of edges of \( K \). In each edge \( E_i \) of \( K \) choose a free arc \( A_i \). Let \( l = V(G) + 1 \) and

\[
\varepsilon_1 = \min \{ d(A_i, A_j) : 1 \leq i < j \leq n \}, \quad \varepsilon_2 = \min \{ \frac{diam(A_i)}{2l} : 1 \leq i \leq n \}.
\]

Let \( 0 < \varepsilon < \min \{ \varepsilon_1, \varepsilon_2 \} \) and \( g_\varepsilon : K \to G \) be an \( \varepsilon \)-map. By dividing \( A_i \) into \( 2l \) subintervals with length \( \frac{diam(A_i)}{2l} \) we get that there is a subinterval \( A'_i \) such that \( g_\varepsilon(A'_i) \) is a free arc of \( G \).
Let $C_1, \ldots, C_p$ be the closures of connected components of $K \setminus (\bigcup_{i=1}^n A'_i)$. Then we have

1. $C_i$ is a star;
2. $G = \bigcup_{i=1}^p g_e(C_i) \cup \bigcup_{i=1}^n g_e(A'_i)$;
3. $g_e(C_i) \cap g_e(C_j) = \emptyset$ for $i \neq j$;
4. if $C_i \cap A'_j \neq \emptyset$, then $g_e(C_i) \cap g_e(A'_j)$ is non-empty and a proper subinterval (may be degenerate) of $g_e(A'_j)$. Moreover, $E(g_e(C_i)) \geq E(C_i)$;
5. $g_e(A'_i) \cap g_e(A'_j) = \emptyset$ for $i \neq j$.

Hence a homeomorphic copy of $K$ can be obtained by shrinking $g_e(C_i)$, $1 \leq i \leq p$, to points. That is, $K \leq G$. 

Continua $M_1$ and $M_2$ are said to be quasi-homeomorphic if $M_1$ is $M_2$-like and $M_2$ is $M_1$-like. It is well known that there are quasi-homeomorphic continua which are not homeomorphic (see for instance [K]). Contrary to this situation we have

Corollary 2.4. Let $G$ and $K$ be graphs. Then $G$ and $K$ are homeomorphic if and only if $G$ and $K$ are quasi-homeomorphic.

Proof. Assume that $G$ and $K$ are quasi-homeomorphic. By Theorem 2.3 and Remark 2.2 we get that $E(K) + B(K) = E(G) + B(G)$. It is easy to say that $G$ and $K$ should be homeomorphic by Theorem 2.3. 

In [MS] the authors show that if a locally connected continuum $M$ is arc-like (circle-like), then $M$ is an arc (a circle). Generalizing this result we have

Theorem 2.5. Let $M$ be a locally connected continuum and $G$ be a graph. Then $M$ is $G$-like if and only if $M$ is a graph and $M \leq G$.

To prove it we need the following simple lemma and the definition of the order of a point in a continuum (see [N], pp. 141–142).

Lemma 2.6. Let $G$, $K$ be graphs. If $K$ is $G$-like, then there is an $\epsilon$-map $f_\epsilon : K \to G$ such that $f_\epsilon(b(K)) \subset b(G)$.

Proof. If there is $b \in b(K)$ such that $f_\epsilon(b) \notin b(G)$ for $\epsilon \to 0$, then the image of some $n$-star (a small closed connected neighbourhood of $b$ with $n = \text{Val}(b)$) under $f_\epsilon$ is an arc. That is, $n$-star ($n \geq 3$) is arc-like. This is impossible by Theorem 2.3. Hence the lemma follows.

Proof of Theorem 2.5. We need to show that if $M$ is $G$-like, then $M$ is a graph.

As $M$ is locally connected, $M$ is path connected. Assume the contrary. That is, $M$ is not a graph. Then there are $n = B(G) + 1$ different points $x_1, \ldots, x_n$ of $M$ such that $\text{Ord}(x_i, M) \geq 3$ ([N], p. 144]). Then there are disjoint graphs $G_i \subset M$, $1 \leq i \leq n$, such that each $G_i$ has at least one branch point and $x_i \in G_i$. Applying Lemma 2.6 we get that $G$ has at least $n$ branch points, a contradiction.

3. A generalization of Burgess’s theorem

A well known result in continuum theory is that if a continuum is both arc-like and circle-like, then $M$ is indecomposable or $2$-indecomposable. In this section we will generalize this result by considering the structure of $G_i$-like ($i = 1, \ldots, m$) continuum $M$. It turns out that in order that $M$ should be $n$-indecomposable for some $n = n(M) \in \mathbb{N}$, $G_i$ ($i = 1, \ldots, m$) must have no common “shape”. To do this we need the following lemma.
Lemma 3.1. Let $T$ be a tree and be an essential sum of the subtrees of $\{T_1, \ldots, T_m\}$ for some $m \in \mathbb{N}$. Then there are at most $\sum_{t \in b(T)} \text{Val}(t)$ elements of $\{T_1, \ldots, T_m\}$ which contain some points of $b(T)$.

Proof. Assume that $d$ is a metric on $T$. For each $b \in b(T)$ let $S(b)$ be the union of edges of $T$ containing $b$ and $e(S(b)) = \{e^1_b, \ldots, e^k_b\}$. Furthermore, let $A = \{T_1, \ldots, T_m\}$. For each $e^i_b$ with $b \in b(T)$ and $1 \leq i \leq k$ choose $T^i_b \in A$ such that

$$d(e^i_b, T^i_b) = \min\{d(e^i_b, S) : S \in A \text{ and } S \text{ contains } b\}.$$ 

We claim that if $S \in A$ and $S$ contains some point of $b(T)$, then $S \subset \bigcup_{i=1}^{k^b} \bigcup_{b \in b(T)} T^i_b$. Assume the contrary. That is, $S \not\subset \bigcup_{i=1}^{k^b} \bigcup_{b \in b(T)} T^i_b$. Then there is $x \in S \cap (T \setminus b(T))$ with $x \not\subset \bigcup_{i=1}^{k^b} \bigcup_{b \in b(T)} T^i_b$. Let $E = [v_1, v_2]$ be the edge of $T$ containing $x$, and without loss of generality we assume that $v_1 \in S \cap b(T)$ and $v_2 = e^j_b$. By the choice of $T^i_b$ we have that $x \in T^i_b$, a contradiction. This proves the claim.

As $T$ is the essential sum of $A$, we have that there are at most $\sum_{t \in b(T)} \text{Val}(t)$ elements of $A$ which contain some points of $b(T)$.

Corollary 3.2. Let $T$ be a tree and an essential sum of the subtrees of $\{T_1, \ldots, T_m\}$ with $m = \sum_{t \in b(T) \cup e(T)} \text{Val}(t)$ for some $k \in \mathbb{N}$. Then there are at least $k$ elements of $\{T_1, \ldots, T_m\}$ which are free arcs of $T$. Furthermore, there are at least $[(k+1)/2]$ pairwise disjoint free arcs from $\{T_1, \ldots, T_m\}$, where $[\ast]$ is the integer part of $\ast$.

Proof. The first conclusion is an immediate consequence of Lemma 3.1. And the second one can be proved easily by induction on $k$.

Note that we will use $\lim \{X, f_i\}$ to denote the inverse limit space of $f_i : X \rightarrow X$, $i \in \mathbb{N}$.

Theorem 3.3. Let $T$ be a tree and $G$ be a graph such that no free arc of $G$ separates $G$. If $M$ is a continuum which is both $T$-like and $G$-like, then $M$ is $n$-indecomposable for some $n \leq n_0 = 2l + \sum_{t \in b(T) \cup e(T)} \text{Val}(t)$, where $l = B(G)$.

Proof. Assume that $M$ is an essential sum of subcontinua $M_1, \ldots, M_{n_0+1}$. Let $M = \lim \{T, f_i\} = \lim \{G, g_i\}$, and $p_i : M \rightarrow T$ and $q_i : M \rightarrow G$ be the $i$-th projection, $i \in \mathbb{N}$. It is easy to see that for $i$ large enough, $T$ is an essential sum of subtrees of $\{p_i(M_1), \ldots, p_i(M_{n_0+1})\}$. By Corollary 3.2 there are at least $l+1$ elements $\{p_i(M_{i_1}), \ldots, p_i(M_{i_{l+1}})\}$ of $\{p_i(M_1), \ldots, p_i(M_{n_0+1})\}$ which are pairwise disjoint free arcs. Hence $M_{i_1}, \ldots, M_{i_{l+1}}$ are pairwise disjoint and each $M_i$ separates $M$.

Thus for $j$ large enough $\{q_j(M_{i_1}), \ldots, q_j(M_{i_{l+1}})\}$ are pairwise disjoint. By the choice of $l$, there is $1 \leq h \leq l+1$ such that $q_j(M_{i_h})$ is a free arc of $G$ for infinitely many $j$. By the assumption on $G$, $q_j(M_{i_h})$ does not separate $G$.

Let $N_1, N_2$ be the two connected components of $\bigcup_{i=1, i \neq h}^{n_0+1} M_i$ and $\epsilon < d(N_1, N_2)$. Choose $j_0$ such that $q_j$ is an $\epsilon$-map for $j \geq j_0$. As $q_j(M_{i_h})$ does not separate $G$, there exist $x \in N_1$ and $y \in N_2$ such that $q_j(x) = q_j(y)$, a contradiction.

Corollary 3.4 (Burgess). If a continuum is both arc-like and circle-like, then $M$ is either indecomposable or the union of two indecomposable subcontinua.

Proof. As $E([0,1]) = 2$ and $B(S^1) = 0$, the corollary follows from Theorem 3.3 immediately.
The following remark and example demonstrate that our result is more general than the result of Burgess. Let $X$ and $Y$ be two topological spaces. A continuous map $f : X \to Y$ is null homotopic provided that $f$ is homotopic to a constant map from $X$ into $Y$.

**Remark 3.5.** Let $G$ be a graph such that no free arc of $G$ separates $G$. If $f : G \to G$ is a surjective map and $f$ is null homotopic, then the inverse limit $M = \lim \{G, f\}$ is $n$-indecomposable for some $n \leq n_0$, where $n_0$ is the number defined in Theorem 3.3.

**Proof.** Let $\tilde{G}$ be the universal cover of $G$. Then $\tilde{G}$ is an infinite tree such that each connected compact subset of $\tilde{G}$ is a finite tree. Let $p : \tilde{G} \to G$ be the covering projection. Since $f$ is null homotopic, there is a lifting $L : G \to \tilde{G}$ with $p \circ L = f$. Put $T = L(G)$. Then $T$ is a finite tree and $p(T) = f(G) = G$. Set $F = L \circ p$. Then $p \circ F = f \circ p$, $F \circ L = L \circ f$ and $F(T) = T$.

Set $\tilde{p} = p|_T : T \to G$, $F' = F|_T : T \to T$, and $L' = L : G \to T$. Let $L_\infty' : M \to \lim \{T, F'\}$, $f_\infty : M \to M$ and $p_\infty' : \lim \{T, F'\} \to M$ be the induced maps (see [N] p. 26). Then $\tilde{p}_\infty' \circ L_\infty' = f_\infty$. Since $f_\infty$ is a homeomorphism, $L_\infty'$ is injective. It is clear that $L_\infty'$ is surjective. Hence $L_\infty'$ is a homeomorphism. Then $M$ is $T$-like and hence $M$ is both $G$-like and $T$-like. By Theorem 3.3, $M$ is $n$-indecomposable for some $n \leq n_0$. $\square$

**Example.** For $m \in \mathbb{N}$ and each $1 \leq i \leq m$, let $K_i$ be the copy of the Knaster’s indecomposable continuum and $p$ be the end point of $K$. Let $M$ be the one point union of $(K_i, p)$, $i = 1, \ldots, m$. Then $K$ is $m$-indecomposable, and $K$ is $m$-od-like and $G$-like, where $G$ is the one point union of $m$ circles.

With the above preparation now we prove the main result of this section. Note that for any finite graphs $G_1, \ldots, G_m$ ($m \in \mathbb{N}$), there are many continua which are $G_i$-like for each $i$, since we can use inverse systems whose terms are $G_i$ (each $G_i$ appears infinitely many times) and arbitrary surjective maps between them.

**Theorem 3.6.** Let $G_i$ ($1 \leq i \leq m$) be graphs. Then in order that each $G_i$-like $(i = 1, \ldots, m)$ continuum $M$ be $n$-indecomposable for some $n = n(M) \in \mathbb{N}$ it is necessary and sufficient that if $K$ is a graph, then $K$ is not $G_i$-like for some integer $i$ with $1 \leq i \leq m$.

**Proof.** (Necessity) It is obvious.

(Sufficiency) It is easy to see that $m \geq 2$. If each $G_i$ contains a simple closed curve, then by Theorem 2.3 $S^1$ is $G_i$-like, $i = 1, \ldots, m$. If each $G_i$ is separated by some free arc, then by Theorem 2.3 $[0, 1]$ is $G_i$-like, $i = 1, \ldots, m$. Hence if there is no graph $K$ which is $G_i$-like ($i = 1, \ldots, m$), then there are $i_0, j_0$ such that $G_{i_0}$ is a tree and $G_{j_0}$ is a graph such that each free arc of $G_{j_0}$ does not separate $G_{j_0}$. According to Theorem 3.3, $M$ is $n$-indecomposable for some $n = n(M) \in \mathbb{N}$. $\square$

The following related problems remain open:

**Question 1.** Let $T$ be a tree and $G$ be a graph such that no free arc of $G$ separates $G$. Let

$$N(T, G) = \{n : M$ is both $T$-like and $G$-like, and is $n$-indecomposable$\}.$$

Is it true that $N(T, G) = \{1, \ldots, n_0\}$ for some $n_0 \leq 2B(G) + \sum_{t \in \mathcal{M}(T)} Val(t)$? If not, determine $N(T, G)$. 


Question 2. Let $T$ be a tree and $G$ be a graph such that no free arc of $G$ separates $G$. Let $M$ be a continuum which is $T$-like. Is it true that if $M$ is $n$-indecomposable for some $n \in N(T, G)$, then $M$ is $G$-like?

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