CHARACTERISTIC CLASSES FOR COMPLEX BUNDLES WITH TRIVIAL REAL REDUCTION

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Abstract. This note concerns itself with a theory of characteristic classes for those complex bundles whose real reductions are trivial.

1

For a topological space \(X\) let \(A_n(X)\) be the set of the complex \(n\)-bundles over \(X\) obtained by furnishing the trivial real bundle \(X \times R^{2n} \to X\) a complex structure. Two bundles \(\xi_0, \xi_1 \in A_n(X)\) are considered to be equivalent if there exists some \(\xi \in A_n(X \times I)\) so that \(\xi | X \times i = \xi_i, i = 0, 1\). Denote by \(B_n(X)\) the set of all such equivalence classes.

It should be noted that our partition on \(A_n(X)\) by the equivalence classes is subtler than the one given by isomorphism classes of complex bundles.

2

Elements of \(B_n(X)\) can be classified within homotopy theory. Let \(CS_n\) be the space of all complex structures \(J\)'s on the \(2n\)-dimensional Euclidean space \(R^{2n}\). The operator \(K: CS_n \times R^{2n} \to CS_n \times R^{2n}\) defined by \(K(J, v) = (J, Jv)\) equips the trivial real vector bundle \(CS_n \times R^{2n} \to CS_n\) a complex structure. Denote by \(\gamma_n\) the resulting complex \(n\)-bundle over \(CS_n\). It will be called the canonical bundle over \(CS_n\).

For two topological spaces \(X, Y\), let \([X, Y]\) be the set of homotopy classes of continuous maps \(X \to Y\).

Proposition 1. For any topological space \(X\) the correspondence \(h: [X, CS_n] \to B_n(X), h([f]) = f^*\gamma_n\), is a bijection.

Indeed the inverse of \(h\) is seen as follows. For a \(\xi \in B_n(X)\) consider the complex structure on \(\xi\) as an \(R\)-linear morphism \(J_\xi: X \times R^{2n} \to X \times R^{2n}\) of the real reduction of \(\xi\). The map \(f_\xi: X \to CS_n\) assigning to each \(x \in X\) the complex structure \(J_\xi | x \times R^{2n} \in CS_n\) is continuous and satisfies \(f_\xi^*\gamma_n = \xi\). It will be called the classifying map of \(\xi\).

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The space $CS_n$ has two connected components, both diffeomorphic to the homogeneous space $SO(2n)/U(n)$. Denote by $CS_n^+$ the component that contains

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{(n copies),}$$

and by $CS_n^-$ the other.

Let $1 + c_1 + \cdots + c_n$ be the total Chern characteristic class for the restricted bundle $\gamma_n | CS_n^+$. We describe $H^*(CS_n^+; Z)$, the integral cohomology algebra of $CS_n^+$, in

**Proposition 2.** The classes $e_i$, $i = 1, \cdots, n-1$, are all divisible by 2. Further if we put $e_1 = \frac{1}{2}e_i$, then $e_1, \cdots, e_{n-1}$ form a simple system of generators for $H^*(CS_n^+; Z)$ [3, p.372] and are subject to the relations

$$R_i : e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^i e_{2i} = 0, \quad i = 1, \cdots, n-1,$$

with the convention $e_k = 0$, $k > n-1$, being understood.

The cohomology algebra of the space $CS_n^+$ has been determined for $Z_2$ coefficients by C. Miller [3], and for $Z_p$ (p a prime) coefficients by A. Borel [1]. The proof of Proposition 2 will be postponed until the final Section 7.

By the first $\left\lfloor \frac{n+1}{2} \right\rfloor - 1$ relations each $e_{2i}$ can be expressed as a polynomial $g_i$ in $e_{\text{odd}}$. For instance the first four such polynomials are given by

$$g_1 = e_1^2,$$
$$g_2 = 2e_1e_3 - e_1^3,$$
$$g_3 = e_3^2 + 2e_1e_5 - 4e_1^2e_3 + 2e_1^4,$$
$$g_4 = 2e_3e_5 + 2e_1e_7 - 6e_1^2e_3^2 + 8e_1^3e_3 - 3e_1^8 - 4e_1^3e_5.$$

Consequently substituting $e_{2i}$ by $g_i$ in the remaining $n - \left\lceil \frac{n+1}{2} \right\rceil$ relations gives rise to equations

$$h_k = 0, \quad k = \left\lfloor \frac{n+1}{2} \right\rfloor, \cdots, n-1,$$

where each $h_k$ is a polynomial in $e_{\text{odd}}$ of degree $4k$. For example when $n = 9$ these polynomials are

$$h_5 = e_1^2 - 2g_2g_3 + 2e_3e_7 - 2g_1g_4,$$
$$h_6 = g_2^2 - 2e_1e_7 + 2g_2g_4,$$
$$h_7 = e_7^2 - 2g_3g_4,$$
$$h_8 = g_4^2.$$

Consider the algebra

$$\Phi = \begin{cases} Z[d_1, d_3, \cdots, d_{n-2}] \otimes \Lambda_Z(v_{2n+1}, v_{2n+3}, \cdots, v_{4n-5}) \text{ when } n \text{ is odd,} \\
Z[d_1, d_3, \cdots, d_{n-1}] \otimes \Lambda_Z(v_{2n-1}, v_{2n+3}, \cdots, v_{4n-5}) \text{ when } n \text{ is even,} \end{cases}$$

the tensor product of the polynomial algebra in $d_{\text{odd}}$ with the exterior algebra in $v_j$. It is graded by

$$\dim(d_i) = 2i \quad \text{and} \quad \dim(v_j) = j.$$

The differential $\delta : \Phi \to \Phi$ of degree 1 given by

$$\delta(d_i) = 0, \delta(v_j) = h_j \frac{d}{d_i} (d_1, d_3, \cdots)$$
furnishes the algebra $\Phi$ with the structure of a differential graded commutative free algebra over $Z$. Moreover since $CS^+_n$ is a symmetric space, Proposition 2 implies

**Proposition 3.** The algebra map $l : (\Phi, \delta) \to (H^*(CS^+_n; Z), \delta = 0)$ defined by $l(d_i) = e_i, l(v_j) = 0$ is the minimal model (over $Z$) for the space $CS^+_n$ (cf. [2 p.158]).

The connected component decomposition $CS_n = CS^+_n \sqcup CS^-_n$ yields a natural partition $[X; CS_n] = [X; CS^+_n] \sqcup [X, CS^-_n]$. Let $B^+_n(X)$ be the $h$-image of $[X; CS^+_n]$ in $B_n(X)$. Since $CS^+_n$ and $CS^-_n$ are mutually diffeomorphic, a theory of characteristic classes on $B^+_n(X)$ corresponds to one on $B^-_n(X)$. So our remaining discussion will focus on the former.

From now on assume that our space $X$ is either a simply connected cell complex or has the rational homotopy type as a product of odd dimensional spheres. Let $(\Phi(X), \delta_X)$ be the rational minimal model of $X$. For a continuous map $f : X \to CS^+_n$, let $\Phi(f)$ be the homotopy class of a minimal model $\Phi \otimes Q \to \Phi(X)$ of $f$. Denote by $[\Phi \otimes Q, \Phi(X)]$ the set of homotopy classes of differential graded algebra maps $\Phi \otimes Q \to \Phi(X)$. Consider the functorial correspondence

$$[X, CS^+_n] \to [\Phi \otimes Q, \Phi(X)]$$

given by $[f] \mapsto \Phi(f)$ [2 p.173].

**Definition.** Let $f : X \to CS^+_n$ be the classifying map of a $\xi \in B^+_n(X)$. The sets

$$d_i(\xi) = \{g(d_i \otimes 1) \in \Phi(X) \mid g \in \Phi(f)\}$$

will be called the primary Chern characteristic sets of $\xi$. The sets

$$v_j(\xi) = \{g(v_j \otimes 1) \in \Phi(X) \mid g \in \Phi(f)\}$$

will be called the secondary Chern characteristic sets of $\xi$. $\square$

Since the forms $d_i$ are closed in $\Phi$, each set $d_i(\xi)$ consists of closed forms and the difference of any two such forms is a coboundary. Therefore passing to cohomology yields an unique class $\{d_i(\xi)\} \in H^{2i}(X; Q)$. The following corollary of Propositions 2 and 3 implies that these classes constitute nothing essentially new

**Proposition 4.** In the rational cohomology $H^*(X; Q)$ the usual Chern characteristic classes $c_1(\xi), \ldots, c_n(\xi)$ of a $\xi \in B^+_n(X)$ can be given in terms of the primary Chern characteristic sets by the formulas

$$c_{2k+1}(\xi) = 2\{d_{2k+1}(\xi)\}; \quad c_{2k}(\xi) = 2g_k(\{d_1(\xi)\}, \{d_3(\xi)\}, \ldots); \quad c_n(\xi) = 0. \quad \Box$$

It is the secondary Chern characteristic sets $v_j(\xi)$ that are of our interest. We observe that some homotopy and cohomology invariants can be extracted from the sets $v_j(\xi)$ even though the forms $v_j \otimes 1$ themselves are not closed in $\Phi \otimes Q$.

**Observation 1.** By rational homotopy theory the forms $v_j \otimes 1$ constitute a basis for $\text{Hom}(\pi_{odd}(CS^+_n), Q)$ and for a continuous map $f : X \to CS^+_n$, the set of induced chain maps

$$\Phi(f) : \Phi \otimes Q \to \Phi(X),$$
module decompositables, agrees with the dual action of the induced homotopy homomorphism
\[ f_* : \pi_* (X) \to \pi_* (CS^+_n). \]
(See [2] p.175, or Lemma 3 in Section 6.) Thus the element
\[ \pi_j (\xi) = v_j (\xi) \] module decompositables
is well defined in \( \text{Hom} (\pi_{\text{odd}} (X); Q) = \pi_{\text{odd}} (X) \otimes Q \).

**Observation 2.** Assume that our space \( X \) has the rational homotopy type as a product of odd-dimensional spheres. Then the minimal model \( \Phi (X) \) agrees with the rational cohomology of \( X \). Thus each set \( v_j (\xi) \) actually consists of elements in \( H^* (X; Q) \).

**Observation 3.** Suppose the bundle \( \xi \in B_n (X) \) with classifying map \( f_\xi \) is such that
\[ d_i (\xi) = \{ 0 \}, i = 1, 3, \ldots \]
(this happens, in particular, when \( X \) is \( 2n - 2 \) connected). Let \( \delta_X \) be the differential of \( \Phi (X) \). Then
\[ \delta_X \Phi (f_\xi) (v_j) = \Phi (f_\xi) \delta v_j = \Phi (f_\xi) h_{\frac{1}{4}+} (d_1, d_3, \ldots) \]
\[ = h_{\frac{1}{4}+} (d_1 (\xi), d_3 (\xi), \ldots) = 0 \]
indicates that \( v_j (\xi) \) consists of closed forms. Similarly one can show that the difference of any two forms in \( v_j (\xi) \) is a coboundary. That is, each set \( v_j (\xi) \) survives to a unique element of \( H^* (X; Q) \).

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The homotopy and cohomology invariants associated with a \( \xi \in B_n^+ (X) \) in the previous section can well be nontrivial even if the usual Chern classes \( c_1 (\xi), \ldots, c_{n-1} (\xi) \) all vanish.

Let \( SO (2n) \) be the special orthogonal group of order \( 2n \), and let \( f_k : SO (2n) \to CS^+_n \), for an integer \( k \in \mathbb{Z} \), be defined by \( f_k (g) = g^k J_0 g^{k'} \), where \( J_0 \in CS^+_n \) is specified in Section 3 and \( ^t \) is the transpose operator. Denote by \( \xi_k \) the induced bundle \( f_k^* \gamma_n \).

**Theorem.** For the sequence of complex \( n \)-bundles \( \{ \xi_k \in B_n^+ (SO (2n)) \mid k \in \mathbb{Z} \} \)
1) the secondary Chern characteristic classes \( \pi_i \) and \( v_i \) are distinctive;
2) the usual Chern characteristic classes \( c_i \) vanish.

Using the grading \( \bigoplus \Phi^r (X) \) of the model \( \Phi (X) \) a graded vector space \( I (X) = \bigoplus I^r (X) \) can be introduced by setting
\[ I^r (X) = \Phi^r (X) / \text{decompositable forms}. \]

For instance it follows from Proposition 3 that

**Lemma 1.**
\[ I (CS^+_n) = \begin{cases} \text{span}_Q \{ e_1, e_3, \ldots, e_{n-1}, v_{2n-1}, v_{2n+3}, \ldots, v_{4n-5} \} & \text{if } n \text{ is even}, \\ \text{span}_Q \{ e_1, e_3, \ldots, e_{n-2}, v_{2n+1}, v_{2n+5}, \ldots, v_{4n-5} \} & \text{if } n \text{ is odd}. \end{cases} \]
For a compact connected Lie group $G$ the minimal model $\Phi(G)$ agrees with $H^*(G; Q)$. From the well known isomorphisms
\[
H^*(U(n); Q) = \Lambda_Q(y_1, y_3, \cdots, y_{2n-1}),
\]
\[
H^*(SO(2n); Q) = \Lambda_Q(x_5, x_7, \cdots, x_{4n-5}, w_{2n-1})
\]
(\text{where the generators } y_i, x_i \text{ and } w_{2n-1} \text{ are primitive with the suffixes indicating their dimensions}) we get

**Lemma 2.**
\[
\begin{align*}
I(U(n)) &= \text{span}_Q \{y_1, y_3, \cdots, y_{2n-1}\}, \\
I(SO(2n)) &= \text{span}_Q \{x_5, x_7, \cdots, x_{4n-5}, w_{2n-1}\}. \quad \square
\end{align*}
\]

A continuous map $f : X \to Y$ induces a well defined homomorphism $I(f) : I(Y) \to I(X)$. We recall from \cite{2} p.175 that

**Lemma 3.** There is a canonical isomorphism $I^r(X) \to \text{Hom}(\pi_r(X), Q)$ which is natural with respect to homomorphisms induced by maps $X \to Y$. \quad \square

For an integer $k \in Z$ let $l_k$ be the self-map of $SO(2n)$ defined by $q \to g^k$. Since the induced algebra map $H^*(SO(2n); Q) \to H^*(SO(2n); Q)$ is given by $l_k^*(x_i) = kx_i$, $l_k^*(w_{2n-1}) = kw_{2n-1}$, we have

**Lemma 4.** The induced endomorphism $I(l_k)$ of $I(SO(2n))$ is multiplication by $k$. \quad \square

Clearly the map $f_1 : SO(2n) \to CS^+_n$ is the projection of the standard fibration
\[
U(n) \subset SO(2n) \to CS^+_n.
\]

Applying the natural transformation $\text{Hom}(_, Q)$ to the homotopy exact sequence of $f_1$ yields the exact sequence of vector spaces
\[
\cdots \to I^r(U(n)) \leftarrow I^r(SO(2n)) \overset{I^r(f_1)}{\leftarrow} I^r(CS^+_n) \leftarrow I^{r-1}(U(n)) \to \cdots.
\]

A dimension comparison discussion based on Lemmas 1 and 2 concludes

**Lemma 5.** The homomorphism $I^r(f_1) : I^r(CS^+_n) \to I^r(SO(2n))$ is
\[
\begin{align*}
1) \text{ an isomorphism for } &r = 2n + 1, 2n + 5, \cdots, 4n - 5 \text{ when } n \text{ is odd, and for } \\
& \phantom{1)} r = 2n + 3, 2n + 7, \cdots, 4n - 5 \text{ when } n \text{ is even}; \\
2) \text{ an injection for } &r = 2n - 1 \text{ when } n \text{ is even.} \quad \square
\end{align*}
\]

**Proof of the Theorem.** Since $f_k = f_1 \circ l_k$, 1) is immediate from Lemmas 4 and 5.

For 2) consider the flag manifold $SO(2n)/T^n$ as the set of orthogonal decompositions of $R^{2n}$ into oriented 2-planes $R^{2n} = L_1 \oplus \cdots \oplus L_n$. Assigning to each $L_1 \oplus \cdots \oplus L_n \in SO(2n)/T^n$ with $\omega_1 \oplus \cdots \oplus \omega_n \in CS^+_n$, where $\omega_i$ is the $\pi$ rotation on $L_i$ with respect to the orientation, yields a map $g : SO(2n)/T^n \to CS^+_n$. In fact $f_1$ factors through the space $SO(2n)/T^n$ in the fashion
\[
\begin{array}{ccc}
SO(2n) & \xrightarrow{q} & SO(2n)/T^n \\
\downarrow & & \downarrow f_1 \\
CS^+_n & \xrightarrow{g} & CS^+_n
\end{array}
\]
where $q$ is the standard projection (for $T^n \subset U(n)$). Since the induced bundle $g^*\gamma_n$ admits a canonical splitting into complex line bundles (this is indicated by
our description of \( g \) we find
\[
g^* c_r(\gamma_n) = \text{the } r\text{th elementary symmetric polynomial}
\]
in some \( x_1, \ldots, x_n \in H^2(SO(2n)/T^n; Q) \).

The proof of 2) is now completed by the facts \( H^2(SO(2n); Q) = 0 \) and \( f_k = f_1 \circ l_k = g \circ q \circ l_k \).

It also follows from Lemmas 2 and 5 that a sequence of complex \( n \)-bundles over each of the spheres
\[
S^r, r = \begin{cases} 
2n + 1, 2n + 5, \ldots, 4n - 5 & \text{when } n \text{ is odd,} \\
2n - 1, 2n + 3, \ldots, 4n - 5 & \text{when } n \text{ is even,}
\end{cases}
\]
with distinctive secondary Chern classes \( v_r \in H^*(S^r; Q) = Q \) (in the sense of Observation 3) can be obtained from the bundles \( \xi_k, k \in \mathbb{Z} \).

This section is devoted to a proof of Proposition 2.

Let \( s_1, \ldots, s_{2n} \) be the standard vector space basis for \( R^{2n} \), and let \( S^{2n-2} \) be the unit sphere in the subspace \( R^{2n-1} \) spanned by \( s_1, \ldots, s_{2n-1} \). The map
\[
\pi : CS_n^+ \to S^{2n-2}, \quad \pi(J) = Js_{2n} \in S^{2n-2}
\]
is a fiber bundle projection whose fiber inclusion over \( s_{2n-1} \in S^{2n-2} \), with respect to the standard splitting \( R^{2n} = \text{span}\{s_1, \ldots, s_{2n-2}\} \oplus \text{span}\{s_{2n-1}, s_{2n}\} \), is
\[
i_n : CS_{n-1}^+ \to CS_n^+, \quad i_n(J') = J' \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The proof proceeds by an induction on \( n \). Let \( \gamma \) be the Hopf line bundle over the 2-sphere \( S^2 \). Since \( CS_2^+ = S^2 \) and since \( \gamma_2 = \gamma \oplus \gamma \), Proposition 2 is true for \( n = 2 \). Assume that it has been proved for \( n - 1 \).

Let \( c_r(\gamma_n) \) be the \( r \)th Chern class of \( \gamma_n \). Since \( H^*(CS_n^+; Z) \) is torsion free and since the real reduction of \( \gamma_n \) is trivial, the 2-divisibility of \( c_r(\gamma_n) \) follows from \( c_r(\gamma) = 0 \mod 2 \).

Consider \( e_r(\gamma_n) = \frac{1}{r!} c_r(\gamma_n) \), \( r = 1, 2, \ldots, n - 1 \). By the naturality of Chern classes with respect to induced bundles we get from \( i_n^* \gamma_n = \gamma_n \oplus \epsilon \) that

1) \( i_n^* c_r(\gamma_n) = c_r(\gamma_{n-1}) \), \( r = 1, 2, \ldots, n - 2 \),

2) \( i_n^* c_{n-1}(\gamma_n) = 0 \),

where \( i_n^* : H^*(CS_n^+; Z) \to H^*(CS_n^+; Z) \) is the induced homomorphism. Combined with the inductive hypothesis 1) indicates

**Lemma 6.** The bundle \( \pi \) has Leray-Hirsch property [3, p.365].

It now follows from 2) that, if we let \( a \in H^{2n-2}(S^{2n-2}; Z) \) be a generator, then

3) \( e_{n-1}(\gamma_n) = k\pi^* a \) for some \( k \in \mathbb{Z} \).

We evaluate the integer \( k \) in

**Lemma 7.** \( k = \pm 1 \).

**Proof.** Let \( \tau \) be the tangent bundle of the base \( S^{2n-2} \). It has Euler class \( \pm 2a \). The induced bundle \( \pi^* \tau \) is a subbundle of \( \gamma_n \) whose orthogonal complement \( (\pi^* \tau)^\perp \) over a \( J \in CS_n^+ \) is the 2-plane spanned by \( s_{2n}, Js_{2n} \in R^{2n} \). Thus \( \gamma_n \) has a ready made
decomposition into the Whitney sum of complex bundles $\pi^*\tau \oplus (\pi^*\tau)^\perp$ in which the second summand is trivial. This implies

$$c_{n-1}(\gamma_n) = c_{n-1}(\pi^*\tau).$$

However the top Chern class $c_{n-1}(\pi^*\tau)$ can be recognized as the Euler class of $\pi^*\tau$, which is $\pm 2\tau^*a$.

The first statement of Proposition 2 has now been proved by Lemmas 6 and 7 (as well as our inductive hypothesis).

Finally since the real reduction of $\gamma_n$ is trivial, the formulas expressing the Pontrjagin classes of $\gamma_n$ in terms of its Chern classes [4, p.177] give rise to the equations

$$c_r(\gamma_n)^2 - 2c_{r-1}(\gamma_n)c_{r+1}(\gamma_n) + \cdots \pm 2c_1(\gamma_n)c_{2r-1}(\gamma_n) \mp 2c_{2r}(\gamma_n) = 0.$$

Dividing both sides by the common divisor 4 yields the relations $R_1 - R_{n-1}$.

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