CHARACTERISTIC CLASSES FOR COMPLEX BUNDLES
WITH TRIVIAL REAL REDUCTION

Duan Haibao

(Communicated by Ralph Cohen)

ABSTRACT. This note concerns itself with a theory of characteristic classes for those complex bundles whose real reductions are trivial.

1

For a topological space $X$ let $A_n(X)$ be the set of the complex $n$-bundles over $X$ obtained by furnishing the trivial real bundle $X \times \mathbb{R}^{2n} \to X$ a complex structure. Two bundles $\xi_0, \xi_1 \in A_n(X)$ are considered to be equivalent if there exists some $\xi \in A_n(X \times I)$ so that $\xi | X \times i = \xi_i$, $i = 0, 1$. Denote by $B_n(X)$ the set of all such equivalence classes.

It should be noted that our partition on $A_n(X)$ by the equivalence classes is subtler than the one given by isomorphism classes of complex bundles.

2

Elements of $B_n(X)$ can be classified within homotopy theory. Let $CS_n$ be the space of all complex structures $J$’s on the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$. The operator $K : CS_n \times \mathbb{R}^{2n} \to CS_n \times \mathbb{R}^{2n}$ defined by $K(J, v) = (J, Jv)$ equips the trivial real vector bundle $CS_n \times \mathbb{R}^{2n} \to CS_n$ a complex structure. Denote by $\gamma_n$ the resulting complex $n$-bundle over $CS_n$. It will be called the canonical bundle over $CS_n$.

For two topological spaces $X, Y$, let $[X, Y]$ be the set of homotopy classes of continuous maps $X \to Y$.

Proposition 1. For any topological space $X$ the correspondence $h : [X, CS_n] \to B_n(X)$, $h([f]) = f^*\gamma_n$, is a bijection.

Indeed the inverse of $h$ is seen as follows. For a $\xi \in B_n(X)$ consider the complex structure on $\xi$ as an $R$-linear morphism $J_\xi : X \times \mathbb{R}^{2n} \to X \times \mathbb{R}^{2n}$ of the real reduction of $\xi$. The map $f_\xi : X \to CS_n$ assigning to each $x \in X$ the complex structure $J_\xi | x \times \mathbb{R}^{2n} \in CS_n$ is continuous and satisfies $f_\xi^*\gamma_n = \xi$. It will be called the classifying map of $\xi$.

Received by the editors September 11, 1997.

2000 Mathematics Subject Classification. Primary 55R40; Secondary 55P62.

Key words and phrases. Vector bundles, characteristic classes, cohomology and rational homotopy theory.

This work was supported by NSFC Project 1977001.
The space $CS_n$ has two connected components, both diffeomorphic to the homogeneous space $SO(2n)/U(n)$. Denote by $CS_n^+$ the component that contains

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(n\text{ copies}),$$

and by $CS_n^-$ the other.

Let $1 + c_1 + \cdots + c_n$ be the total Chern characteristic class for the restricted bundle $\gamma_n | CS_n^+$. We describe $H^*(CS_n^+; \mathbb{Z})$, the integral cohomology algebra of $CS_n^+$, in

**Proposition 2.** The classes $c_i$, $i = 1, \cdots, n-1$, are all divisible by 2. Further if we put $c_i = \frac{1}{2}c_i$, then $e_1, \cdots, e_{n-1}$ form a simple system of generators for $H^*(CS_n^+; \mathbb{Z})$ [3, p.372] and are subject to the relations

$$R_i : e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^i e_{2i} = 0, \quad i = 1, \cdots, n-1,$$

with the convention $e_k = 0, k > n - 1$, being understood.

The cohomology algebra of the space $CS_n^+$ has been determined for $\mathbb{Z}_2$ coefficients by C. Miller [3], and for $\mathbb{Z}_p$ ($p$ a prime) coefficients by A. Borel [1]. The proof of Proposition 2 will be postponed until the final Section 7.

By the first $[\frac{n+1}{2}] - 1$ relations each $e_{2i}$ can be expressed as a polynomial $g_i$ in $e_{\text{odd}}$. For instance the first four such polynomials are given by

$$g_1 = e_1^2,$$

$$g_2 = 2e_1 e_3 - e_1^3,$$

$$g_3 = e_3^2 + 2e_1 e_5 - 4e_1^3 e_3 + 2e_1^6,$$

$$g_4 = 2e_3 e_5 + 2e_1 e_7 - 6e_1^2 e_3^2 + 8e_1^5 e_3 - 3e_1^8 - 4e_1^3 e_5.$$

Consequently substituting $e_{2i}$ by $g_i$ in the remaining $n - [\frac{n+1}{2}]$ relations gives rise to equations

$$h_k = 0, k = \left[ \frac{n+1}{2} \right], \cdots, n-1,$$

where each $h_k$ is a polynomial in $e_{\text{odd}}$ of degree $4k$. For example when $n = 9$ these polynomials are

$$h_5 = e_5^2 - 2g_2 g_3 + 2e_3 e_7 - 2g_1 g_4,$$

$$h_6 = g_3^2 - 2e_5 e_7 + 2g_2 g_4,$$

$$h_7 = e_7^2 - 2g_3 g_4,$$

$$h_8 = g_4^2.$$

Consider the algebra

$$\Phi = \left\{ \begin{array}{ll} Z[d_1, d_3, \cdots, d_{n-2}] \otimes \Lambda_{\mathbb{Z}}(v_{2n+1}, v_{2n+5}, \cdots, v_{4n-5}) & \text{when } n \text{ is odd}, \\
Z[d_1, d_3, \cdots, d_{n-1}] \otimes \Lambda_{\mathbb{Z}}(v_{2n-1}, v_{2n+3}, \cdots, v_{4n-5}) & \text{when } n \text{ is even}, \end{array} \right.$$

the tensor product of the polynomial algebra in $d_{\text{odd}}$ with the exterior algebra in $v_j$. It is graded by

$$\dim(d_i) = 2i \quad \text{and} \quad \dim(v_j) = j.$$ 

The differential $\delta : \Phi \to \Phi$ of degree 1 given by

$$\delta(d_i) = 0, \delta(v_j) = h_{\frac{j+i}{4}}(d_1, d_3, \cdots)$$
furnishes the algebra $\Phi$ with the structure of a differential graded commutative free algebra over $\mathbb{Z}$. Moreover since $CS_n^+$ is a symmetric space, Proposition 2 implies

**Proposition 3.** The algebra map $l : (\Phi, \delta) \to (H^*(CS_n^+; \mathbb{Z}), \delta = 0)$ defined by $l(d_i) = e_i, l(v_j) = 0$ is the minimal model (over $\mathbb{Z}$) for the space $CS_n^+$ (cf. [2, p.158]).

The connected component decomposition $CS_n = CS_n^+ \sqcup CS_n^-$ yields a natural partition $[X; CS_n] = [X, CS_n^+] \sqcup [X, CS_n^-]$. Let $B^+_n(X)$ be the $h$-image of $[X; CS_n^+]$ in $B_n(X)$. Since $CS_n^+$ and $CS_n^-$ are mutually diffeomorphic, a theory of characteristic classes on $B^+_n(X)$ corresponds to one on $B^-_n(X)$. So our remaining discussion will focus on the former.

From now on assume that our space $X$ is either a simply connected cell complex or has the rational homotopy type as a product of odd dimensional spheres. Let $(\Phi(X), \delta_X)$ be the rational minimal model of $X$. For a continuous map $f : X \to CS_n^+$, let $\Phi(f)$ be the homotopy class of a minimal model $\Phi \otimes \mathbb{Q} \to \Phi(X)$ of $f$. Denote by $[\Phi \otimes \mathbb{Q}, \Phi(X)]$ the set of homotopy classes of differential graded algebra maps $\Phi \otimes \mathbb{Q} \to \Phi(X)$. Consider the functorial correspondence

$$[X, CS_n^+] \to [\Phi \otimes \mathbb{Q}, \Phi(X)]$$

given by $[f] \to \Phi(f)$ [2, p.173].

**Definition.** Let $f : X \to CS_n^+$ be the classifying map of a $\xi \in B^+_n(X)$. The sets

$$d_i(\xi) = \{g(d_i \otimes 1) \in \Phi(X) \mid g \in \Phi(f)\}$$

will be called the primary Chern characteristic sets of $\xi$. The sets

$$v_j(\xi) = \{g(v_j \otimes 1) \in \Phi(X) \mid g \in \Phi(f)\}$$

will be called the secondary Chern characteristic sets of $\xi$.

Since the forms $d_i$ are closed in $\Phi$, each set $d_i(\xi)$ consists of closed forms and the difference of any two such forms is a coboundary. Therefore passing to cohomology yields an unique class $\{d_i(\xi)\} \in H^{2i}(X; \mathbb{Q})$. The following corollary of Propositions 2 and 3 implies that these classes constitute nothing essentially new

**Proposition 4.** In the rational cohomology $H^*(X; \mathbb{Q})$ the usual Chern characteristic classes $c_1(\xi), \ldots, c_n(\xi)$ of a $\xi \in B^+_n(X)$ can be given in terms of the primary Chern characteristic sets by the formulas

$$c_{2k+1}(\xi) = 2\{d_{2k+1}(\xi)\}; \quad c_{2k}(\xi) = 2g_k(\{d_1(\xi)\}, \{d_3(\xi)\}, \ldots); \quad c_n(\xi) = 0.$$  

It is the secondary Chern characteristic sets $v_j(\xi)$ that are of our interest. We observe that some homotopy and cohomology invariants can be extracted from the sets $v_j(\xi)$ even though the forms $v_j \otimes 1$ themselves are not closed in $\Phi \otimes \mathbb{Q}$.

**Observation 1.** By rational homotopy theory the forms $v_j \otimes 1$ constitute a basis for $\text{Hom}(\pi_{\text{odd}}(CS_n^+), \mathbb{Q})$ and for a continuous map $f : X \to CS_n^+$, the set of induced chain maps

$$\Phi(f) : \Phi \otimes \mathbb{Q} \to \Phi(X),$$
module decompositables, agrees with the dual action of the induced homotopy homomorphism

\[ f_* : \pi_*(X) \rightarrow \pi_*(CS_n^+) \]

(See [2, p.175], or Lemma 3 in Section 6.) Thus the element

\[ \pi_j(\xi) = v_j(\xi) \]

is well defined in \( \text{Hom}(\pi_{\text{odd}}(X); Q) = \pi_{\text{odd}}(X) \otimes Q \).

**Observation 2.** Assume that our space \( X \) has the rational homotopy type as a product of odd-dimensional spheres. Then the minimal model \( \Phi(X) \) agrees with the rational cohomology of \( X \). Thus each set \( v_j(\xi) \) actually consists of elements in \( H^*(X; Q) \).

**Observation 3.** Suppose the bundle \( \xi \in B_n(X) \) with classifying map \( f_\xi \) is such that

\[ d_i(\xi) = \{0\}, \quad i = 1, 3, \cdots \]

(this happens, in particular, when \( X \) is \( 2n - 2 \) connected). Let \( \delta_X \) be the differential of \( \Phi(X) \). Then

\[ \delta_X \Phi(f_\xi)(v_j) = \Phi(f_\xi) \delta v_j = \Phi(f_\xi) h_{2n+1}(d_1, d_3, \cdots) = 0 \]

indicates that \( v_j(\xi) \) consists of closed forms. Similarly one can show that the difference of any two forms in \( v_j(\xi) \) is a coboundary. That is, each set \( v_j(\xi) \) survives to a unique element of \( H^*(X; Q) \).

6

The homotopy and cohomology invariants associated with a \( \xi \in B^+_n(X) \) in the previous section can well be nontrivial even if the usual Chern classes \( c_1(\xi), \cdots, c_{n-1}(\xi) \) all vanish.

Let \( SO(2n) \) be the special orthogonal group of order \( 2n \), and let \( f_k : SO(2n) \rightarrow CS_n^+ \), for an integer \( k \in \mathbb{Z} \), be defined by \( f_k(g) = g^k J_0 g^{-k} \), where \( J_0 \in CS_n^+ \) is specified in Section 3 and \( ' \) is the transpose operator. Denote by \( \xi_k \) the induced bundle \( f_k^* \gamma_n \).

**Theorem.** For the sequence of complex \( n \)-bundles \( \{\xi_k \in B^+_n(SO(2n)) \mid k \in \mathbb{Z}\} \)

1) the secondary Chern characteristic classes \( \pi_i \) and \( v_i \) are distinctive;
2) the usual Chern characteristic classes \( c_i \) vanish.

Using the grading \( \bigoplus \Phi^r(X) \) of the model \( \Phi(X) \) a graded vector space \( I(X) = \bigoplus I^r(X) \) can be introduced by setting

\[ I^r(X) = \Phi^r(X) / \text{decompositional forms} \]

For instance it follows from Proposition 3 that

**Lemma 1.**

\[ I(CS_n^+) = \begin{cases} \text{span}_Q \{e_1, e_3, \cdots, e_{n-1}, v_{2n-1}, v_{2n+3}, \cdots, v_{4n-5}\} & \text{if } n \text{ is even} \\
\text{span}_Q \{e_1, e_3, \cdots, e_{n-2}, v_{2n+1}, v_{2n+5}, \cdots, v_{4n-5}\} & \text{if } n \text{ is odd} \end{cases} \]

\[ \square \]
For a compact connected Lie group $G$ the minimal model $\Phi(G)$ agrees with $H^*(G; Q)$. From the well known isomorphisms

\[
H^*(U(n); Q) = \Lambda_Q(y_1, y_3, \cdots, y_{2n-1}), \\
H^*(SO(2n); Q) = \Lambda_Q(x_5, x_7, \cdots, x_{4n-5}, w_{2n-1})
\]

(where the generators $y_i, x_i$ and $w_{2n-1}$ are primitive with the suffixes indicating their dimensions) we get

**Lemma 2.**

\[
I(U(n)) = \text{span}_Q\{y_1, y_3, \cdots, y_{2n-1}\}, \\
I(SO(2n)) = \text{span}_Q\{x_5, x_7, \cdots, x_{4n-5}, w_{2n-1}\}.
\]

A continuous map $f : X \to Y$ induces a well defined homomorphism $I(f) : I(Y) \to I(X)$. We recall from [2, p.175] that

**Lemma 3.** There is a canonical isomorphism $I^r(X) \to \text{Hom}(\pi_r(X), Q)$ which is natural with respect to homomorphisms induced by maps $X \to Y$. □

For an integer $k \in \mathbb{Z}$ let $l_k$ be the self-map of $SO(2n)$ defined by $q \to g^k$. Since the induced algebra map $H^*(SO(2n); Q) \to H^*(SO(2n); Q)$ is given by $l_k^* (x_i) = kx_i, l_k^* (w_{2n-1}) = kw_{2n-1},$ we have

**Lemma 4.** The induced endomorphism $I(l_k)$ of $I(SO(2n))$ is multiplication by $k$. □

Clearly the map $f_1 : SO(2n) \to CS^+_n$ is the projection of the standard fibration

\[
U(n) \subset SO(2n) \to CS^+_n.
\]

Applying the natural transformation $\text{Hom}(\_ ; Q)$ to the homotopy exact sequence of $f_1$ yields the exact sequence of vector spaces

\[
\cdots \leftarrow I^r(U(n)) \leftarrow I^r(SO(2n)) \xrightarrow{I^r(f_1)} I^r(CS^+_n) \leftarrow I^{r-1}(U(n)) \leftarrow \cdots.
\]

A dimension comparison discussion based on Lemmas 1 and 2 concludes

**Lemma 5.** The homomorphism $I^r(f_1) : I^r(CS^+_n) \to I^r(SO(2n))$ is

1) an isomorphism for $r = 2n + 1, 2n + 5, \cdots, 4n - 5$ when $n$ is odd, and for $r = 2n + 3, 2n + 7, \cdots, 4n - 5$ when $n$ is even;

2) an injection for $r = 2n - 1$ when $n$ is even. □

**Proof of the Theorem.** Since $f_k = f_1 \circ l_k, \ 1)$ is immediate from Lemmas 4 and 5.

For 2) consider the flag manifold $SO(2n)/T^n$ as the set of orthogonal decompositions of $R^{2n}$ into oriented 2-planes $R^{2n} = L_1 \oplus \cdots \oplus L_n$. Assigning to each $L_1 \oplus \cdots \oplus L_n \in SO(2n)/T^n$ with $\omega_1 \oplus \cdots \oplus \omega_n \in CS^+_n$, where $\omega_i$ is the $\frac{\pi}{2}$ rotation on $L_i$ with respect to the orientation, yields a map $g : SO(2n)/T^n \to CS^+_n$. In fact $f_1$ factors through the space $SO(2n)/T^n$ in the fashion

\[
\begin{array}{ccc}
SO(2n) & \xrightarrow{q} & SO(2n)/T^n \\
\downarrow & \searrow & \downarrow \xrightarrow{f_1} \\
CS^+_n & \searrow & CS^+_n
\end{array}
\]

where $q$ is the standard projection (for $T^n \subset U(n)$). Since the induced bundle $g^*\gamma_n$ admits a canonical splitting into complex line bundles (this is indicated by
our description of \( g \) we find
\[
g^* c_r(\gamma_n) = \text{the } r\text{th elementary symmetric polynomial in some } x_1, \cdots, x_n \in H^2(SO(2n)/T^n; Q).
\]
The proof of 2) is now completed by the facts \( H^2(SO(2n); Q) = 0 \) and \( f_k = f_1 \circ l_k = g \circ q \circ l_k \).

It also follows from Lemmas 2 and 5 that a sequence of complex \( n \)-bundles over each of the spheres
\[
S^r_n, r = \begin{cases}
2n + 1, 2n + 5, \cdots, 4n - 5 & \text{when } n \text{ is odd},
2n - 1, 2n + 3, \cdots, 4n - 5 & \text{when } n \text{ is even},
\end{cases}
\]
with distinctive secondary Chern classes \( v_r \in H^*(S^r_n; Q) = Q \) (in the sense of Observation 3) can be obtained from the bundles \( \xi_k, k \in Z \).

7

This section is devoted to a proof of Proposition 2.

Let \( s_1, \cdots, s_{2n} \) be the standard vector space basis for \( R^{2n} \), and let \( S^{2n-2} \) be the unit sphere in the subspace \( R^{2n-1} \) spanned by \( s_1, \cdots, s_{2n-1} \). The map
\[
\pi : CS^+_{2n} \to S^{2n-2}, \quad \pi(J) = J s_{2n} \in S^{2n-2}
\]
is a fiber bundle projection whose fiber inclusion over \( s_{2n-1} \in S^{2n-2} \), with respect to the standard splitting \( R^{2n} = \text{span}\{s_1, \cdots, s_{2n-2}\} \oplus \text{span}\{s_{2n-1}, s_{2n}\} \), is
\[
i_n : CS^+_{n-1} \to CS^+_{n}, \quad i_n(J') = J' \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The proof proceeds by an induction on \( n \). Let \( \gamma \) be the Hopf line bundle over the 2-sphere \( S^2 \). Since \( CS^+_2 = S^2 \) and since \( \gamma_2 = \gamma \oplus \gamma \), Proposition 2 is true for \( n = 2 \). Assume that it has been proved for \( n = 1 \).

Let \( c_r(\gamma_n) \) be the \( r \)th Chern class of \( \gamma_n \). Since \( H^*(CS^+_n; Z) \) is torsion free and since the real reduction of \( \gamma_n \) is trivial, the 2-divisibility of \( c_r(\gamma_n) \) follows from \( c_r(\gamma_n) \equiv 0 \mod 2 \).

Consider \( e_r(\gamma_n) = \frac{1}{2} c_r(\gamma_n), r = 1, 2, \cdots, n-1 \). By the naturality of Chern classes with respect to induced bundles we get from \( i_n^* \gamma_n = \gamma_{n-1} \oplus e \) that
\[
1) i_n^* c_r(\gamma_n) = c_r(\gamma_{n-1}), r = 1, 2, \cdots, n-2,
2) i_n^* (c_{n-1}(\gamma_n)) = 0,
\]
where \( i_n^* : H^*(CS^+_n; Z) \to H^*(CS^+_{n-1}; Z) \) is the induced homomorphism. Combined with the inductive hypothesis 1) indicates

Lemma 6. The bundle \( \pi \) has Leray-Hirsch property [5, p.365].

It now follows from 2) that, if we let \( a \in H^{2n-2}(S^{2n-2}; Z) \) be a generator, then
\[
e_{n-1}(\gamma_n) = k \pi^* a \text{ for some } k \in Z.
\]
We evaluate the integer \( k \) in

Lemma 7. \( k = \pm 1 \).

Proof. Let \( \tau \) be the tangent bundle of the base \( S^{2n-2} \). It has Euler class \( \pm 2a \). The induced bundle \( \pi^* \tau \) is a subbundle of \( \gamma_n \) whose orthogonal complement \( (\pi^* \tau)^{\perp} \) over a \( J \in CS^+_n \) is the 2-plane spanned by \( s_{2n}, J s_{2n} \in R^{2n} \). Thus \( \gamma_n \) has a ready made
decomposition into the Whitney sum of complex bundles $\pi^*\tau \oplus (\pi^*\tau)^\perp$ in which the second summand is trivial. This implies

$$c_{n-1}(\gamma_n) = c_{n-1}(\pi^*\tau).$$

However the top Chern class $c_{n-1}(\pi^*\tau)$ can be recognized as the Euler class of $\pi^*\tau$, which is $\pm 2\pi^*a$.

The first statement of Proposition 2 has now been proved by Lemmas 6 and 7 (as well as our inductive hypothesis).

Finally since the real reduction of $\gamma_n$ is trivial, the formulas expressing the Pontrjagin classes of $\gamma_n$ in terms of its Chern classes [4, p.177] give rise to the equations

$$c_r(\gamma_n)^2 - 2c_{r-1}(\gamma_n)c_{r+1}(\gamma_n) + \cdots \pm 2c_1(\gamma_n)c_{2r-1}(\gamma_n) \mp 2c_{2r}(\gamma_n) = 0.$$

Dividing both sides by the common divisor 4 yields the relations $R_1 - R_{n-1}$.

References


Department of Mathematics, Peking University, Beijing 100871, People’s Republic of China

E-mail address: dhb0@xx0.math.pku.edu.cn

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use