A TOPOLOGICAL PROPERTY OF INTEGRABLE SYSTEMS

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Abstract. If we are given \( n \) real-valued smooth functions on \( \mathbb{R}^{2n} \) which are in involution, then, under some mild hypotheses, the subset of \( \mathbb{R}^{2n} \) where these functions are linearly independent is not simply connected.

A system of \( n \) smooth functions \( f_i : \mathbb{R}^{2n} \to \mathbb{R} \), \( 1 \leq i \leq n \), is said to be in involution if the Poisson bracket of any two of them is zero. The classical topological property of systems in involution is given by the following theorem of Arnold:

**Theorem** ([1]). Let \( f_i : \mathbb{R}^{2n} \to \mathbb{R} \), \( 1 \leq i \leq n \), be \( n \) smooth functions in involution and let \( c \in \mathbb{R}^n \) be a regular value of the moment map \( F := (f_1, \ldots, f_n) : \mathbb{R}^{2n} \to \mathbb{R}^n \).

A compact, connected component of \( F^{-1}(c) \) is a Lagrangian torus in \( \mathbb{R}^{2n} \).

In this note we show how a theorem of Viterbo (see [3]), which states that the Maslov class of an embedded Lagrangian torus in \( \mathbb{R}^{2n} \) does not vanish, implies a second topological property for systems of functions in involution.

**Theorem.** Let \( f_i : \mathbb{R}^{2n} \to \mathbb{R} \), \( 1 \leq i \leq n \), be \( n \) smooth functions in involution. If for some regular value \( c \in \mathbb{R}^n \) of the moment map \( F : \mathbb{R}^{2n} \to \mathbb{R}^n \) the set \( F^{-1}(c) \) contains a compact, connected component, then the set of points in \( \mathbb{R}^{2n} \) where \( df_1 \wedge df_2 \wedge \cdots \wedge df_n \neq 0 \) is not simply connected.

**Example** (The harmonic oscillator). Consider the Hamiltonian

\[
H(q_1, q_2, p_1, p_2) := q_1^2 + q_2^2 + p_1^2 + p_2^2
\]

for a harmonic oscillator. The two integrals of motion \( f_1(q, p) := q_1^2 + p_1^2 \) and \( f_2(q, p) := q_2^2 + p_2^2 \) Poisson commute and are independent everywhere except at the union of the two-dimensional planes \( q_1 = p_1 = 0 \) and \( q_2 = p_2 = 0 \). Using the radial projection onto the unit sphere, we see that the set where \( df_1 \wedge df_2 \neq 0 \) is homotopic to the 3-sphere with two fibers of the Hopf fibration deleted. We conclude that this set is not simply connected.

**Proof of the theorem.** By Arnold’s theorem, a compact connected component of \( F^{-1}(c) \) is an embedded Lagrangian torus. We will see that if it were possible to embed this torus into some open, simply connected set \( V \subset \mathbb{R}^{2n} \) where \( F \) has no
singular points, its Maslov class would vanish. This contradicts Viterbo’s theorem (3).

Recall that the Maslov class of a Lagrangian submanifold of $\mathbb{R}^{2n}$ is defined as the pullback by the Gauss map of the generator $\mu$ of the first cohomology class of the Grassmannian of Lagrangian planes. If a Lagrangian torus in $F^{-1}(c)$ is contained in a set $V$ without singular points, we may extend its Gauss map to the whole of $V$ by assigning to a point $x \in V$ the Lagrangian subspace spanned by the Hamiltonian vector fields of the functions $f_i$, $i = 1, \ldots, n$, at the point $x$. Since $V$ is simply connected, the pullback of the class $\mu$ to $V$ under this map is trivial and so is its restriction to $F^{-1}(c)$.

**Remark.** This simple result and its proof were suggested by a theorem of Ehresmann and Reeb (2) on the topology of the leaves of foliations in $\mathbb{R}^n$.

**References**


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