TILTING UP ITERATED TILTED ALGEBRAS

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Abstract. We show that, if $A$ is a representation-finite iterated tilted algebra of euclidean type $Q$, then there exist a sequence of algebras $A = A_0, A_1, A_2, \ldots, A_m$, and a sequence of modules $T^{(i)}_{A_i}$, where $0 \leq i < m$, such that each $T^{(i)}_{A_i}$ is an APR-tilting $A_i$-module, or an APR-cotilting $A_i$-module, $\text{End}_{A_i}(T^{(i)}_{A_i}) = A_{i+1}$ and $A_m$ is tilted representation-finite.

1. Introduction

Throughout this paper, we let $k$ denote a fixed algebraically closed field. By algebra is always meant a finite dimensional associative $k$-algebra with an identity, which we assume moreover to be basic and connected, and by module is meant a finitely generated right $A$-module. Tilting theory is by now an established tool in the representation theory of algebras. In particular, the classes of tilted and iterated tilted algebras, introduced by means of the tilting process, were very useful, for instance, in the classification of the self-injective algebras of polynomial growth (see [1], [7], [11]). We recall their definition. Let $Q$ be a finite, connected quiver without oriented cycles and let $kQ$ denote the path algebra of $Q$. An algebra $A$ is said to be iterated tilted of type $Q$ if there exists a going-up tilting series from $A$ to $kQ$, that is, a sequence of algebras $A = A_0, A_1, \ldots, A_m = kQ$ and a sequence of tilting modules $T^{(i)}_{A_i}$, where $0 \leq i < m$, such that $\text{End}_{A_i}(T^{(i)}_{A_i}) = A_{i+1}$ and each $T^{(i)}_{A_i}$ is separating, that is, such that each indecomposable $A_i$-module $M$ satisfies either $\text{Hom}_{A_i}(T^{(i)}_{A_i}, M) = 0$ or $\text{Ext}^1_{A_i}(T^{(i)}_{A_i}, M) = 0$ (see [2]). This implies that, if $A_i$ is representation-finite, then so is $A_j$ for each $0 \leq j \leq i$. If $m \leq 1$, then $A$ is said to be a tilted algebra of type $Q$.

In order to study the representation-finite iterated tilted algebras of euclidean type, O. Roldán (private communication) has formulated the following conjecture which would allow to reduce their study to that of the representation-finite tilted algebras of the same type. Let $Q$ be a euclidean quiver, and $A$ be a representation-finite iterated tilted algebra of type $Q$. The conjecture states that there exists a sequence of tilts $A = A_0, A_1, \ldots, A_{m-1}, A_m = kQ$ as in the definition above, but with $A_{m-1}$ representation-finite (6.5). We show that the conjecture, in this form, is not true (see example (3.4) (a)), so that separating tilting modules do not suffice to reduce the study of iterated tilted algebras of euclidean type.
to that of representation-finite tilted algebras of the same type. The objective of this paper is to show that the use of both separating tilting and separating cotilting modules (even, of a particularly nice type) will achieve that goal. Namely, replacing the separating tilting modules by APR-tilting or APR-cotilting modules (for the definitions, see (2.1)), one obtains the following statement.

**Theorem.** Let \( Q \) be a euclidean quiver without oriented cycles, and \( A \) be a representation-finite iterated tilted algebra of type \( Q \). Then there exist a sequence of algebras \( A = A_0, A_1, \ldots, A_m \), and a sequence of modules \( T^{(i)}_A \), where \( 0 \leq i < m \), such that each \( T^{(i)}_A \) is an APR-tilting \( A_i \)-module, or an APR-cotilting \( A_i \)-module, \( \text{End} T^{(i)}_A = A_i + 1 \) and \( A_m \) is tilted representation-finite.

Further, we show that this theorem is not true if one allows \( Q \) to be a wild quiver instead of a euclidean quiver (see Example (3.4) (b)).

The case where the underlying graph of \( Q \) is the euclidean diagram \( \tilde{A}_n \) was considered by the third author in \([12], [13]\), using the classification of the tilted and iterated tilted algebras of type \( \tilde{A}_n \) (see \([3], [10]\)). The purpose of the present paper is to provide a proof for the general case that does not use any classification result.

The paper is organised as follows. In section 2, after setting the notation and briefly recalling some definitions and results that will be needed in the sequel, we prove a necessary and sufficient condition for an iterated tilted algebra of euclidean type to be representation-finite, then a lemma on almost complete tilting modules over tame hereditary algebras. Section 3 consists of two lemmata followed by the proof of our theorem and ends with the aforementioned examples.

2. Preliminaries

2.1. **Notation.** Let \( A \) be a finite dimensional basic and connected \( k \)-algebra. We denote by \( Q_A \) the ordinary quiver of \( A \), and by \( (Q_A)_0 \) the set of points of \( Q_A \). For a point \( a \in (Q_A)_0 \), we denote by \( S(a) \) the corresponding simple \( A \)-module, and by \( P(a) \) the projective cover of \( S(a) \). We use freely, and without further reference, facts about the module category \( \text{mod} A \) and the Auslander-Reiten translations \( \tau_A = D \text{Tr} \) and \( \tau^{-1}_A = \text{Tr} D \), as in \([4], [5]\).

An \( A \)-module \( T \) is called a tilting (or cotilting) module if \( \text{pd} T_A \leq 1 \) (or \( \text{id} T_A \leq 1 \), respectively), \( \text{Ext}^1_A(T, T) = 0 \) and the number of isomorphism classes of indecomposable summands of \( T \) equals the number of points in \( Q_A \). Given a tilting module \( T_A \), there exists a close connection between the representation theories of \( A \) and of \( \text{End} T_A \) : this connection is known as tilting theory, and we refer the reader to \([1], [7]\) for the details. A tilting module \( T_A \) is called separating if, for each indecomposable \( A \)-module \( M \), we have either \( \text{Hom}_A(T, M) = 0 \) or \( \text{Ext}^1_A(T, M) = 0 \). If \( T_A \) is a separating tilting module, then the almost split sequences in \( \text{mod} A \) are totally determined by those in \( \text{mod} \text{End} T_A \). An example of separating tilting module is provided by the so-called APR-tilting modules: let \( P(a)_A \) be a simple projective module which is not injective; then the module \( \tau^{-1}_A P(a) \oplus \bigoplus_{b \neq a} P(b) \) is the corresponding APR-tilting module. Separating cotilting modules and APR-cotilting modules are defined dually.

2.2. **The derived category.** Let \( A \) be an algebra. We denote by \( D^b(\text{mod} A) \) the derived category of bounded complexes of \( A \)-modules. The study of \( D^b(\text{mod} A) \) is
especially fruitful if $A$ is an iterated tilted algebra, as is seen from the following theorem \[\text{[7]}\] (IV.5.4), p. 176.

**Theorem.** Let $Q$ be a finite connected quiver without oriented cycles. An algebra $A$ is iterated tilted of type $Q$ if and only if there exists an equivalence of triangulated categories $D^b(\text{mod} A) \cong D^b(\text{mod} kQ)$.

Further, the structure of $D^b(\text{mod} kQ)$ is known \[\text{[7]}\] (I.5.5) p. 54. If, in particular, $Q$ is a euclidean quiver, then the Auslander-Reiten quiver of $kQ$ consists of a component $\mathcal{P}$ containing the projective $kQ$-modules, a component $\mathcal{Q}$ containing the injective $kQ$-modules, and a stable tubular family $\mathcal{R} = (\mathcal{R}_\lambda)_{\lambda \in \mathcal{P}(k)}$ separating $\mathcal{P}$ from $\mathcal{Q}$ (see \[\text{[6]}, \text{[9]}\]). The quiver of $D^b(\text{mod} kQ)$ may then be visualised as shown in Figure 1, where each $\mathcal{P}[i]$ (or $\mathcal{Q}[i], \mathcal{R}[i]$) is a copy of $\mathcal{P}$ (or $\mathcal{Q}, \mathcal{R}$, respectively) indexed by $i \in \mathbb{Z}$. The components $\mathcal{C}[i]$ are referred to as the **transjective components**, and the $\mathcal{R}[i]$ as the **regular components**.

Let $A$ be an iterated tilted algebra of type $Q$. The image of the module $A_A$ under the composition of the embedding of mod $A$ inside $D^b(\text{mod} A)$ with the triangle-equivalence $F : D^b(\text{mod} A) \cong D^b(\text{mod} kQ)$ is a tilting complex $T^\bullet = F(A_A)$ (in the sense of \[\text{[5]}\]) which we call the **standard complex**. For a point $a \in (Q_A)_0$, we write $T^\bullet(a) = FP(a)$. Thus, $T^\bullet = \bigoplus_{a \in (Q_A)_0} T^\bullet(a)$.

**2.3. Proposition.** Let $A$ be an iterated tilted algebra of euclidean type $Q$, and let $F : D^b(\text{mod} A) \to D^b(\text{mod} kQ)$ be a triangle-equivalence. Then $A$ is representation-finite if and only if there exist two points $a \neq b$ in $Q_A$, and two integers $i_a \neq i_b$ such that $T^\bullet(a) \in \mathcal{C}[i_a]$ and $T^\bullet(b) \in \mathcal{C}[i_b]$.

**Proof.** Since the sufficiency is just part (a) of \[\text{[7]}\] (IV.7.1), p. 188, we only show the necessity. We assume that the stated condition does not hold, and prove that this implies that $A$ is representation-infinite.

We first notice that there exists at least one point $a \in (Q_A)_0$ such that $T^\bullet(a)$ lies in a transjective component: indeed, if this is not the case, then the indecomposable summands $T^\bullet(x)$ of the standard complex $T^\bullet$ all belong to the regular components, which would imply that the vectors $\dim T^\bullet(x)$ are linearly dependent, a contradiction to the fact that they form a basis in $K_0(D^b(\text{mod} kQ))$, by \[\text{[7]}\] (III.1.2), p. 96. Therefore the assumption that the stated condition does not hold implies that the indecomposable summands of the standard complex lie in at most one transjective component $\mathcal{C}[i]$, say. The standard complex $T^\bullet$ decomposes thus as $T^\bullet = T'^\bullet \oplus T''^\bullet$, where $T'^\bullet \in \mathcal{C}[i]$ is non-zero, while the indecomposable summands of $T''^\bullet$ are all regular.

Let $X^\bullet \in \mathcal{R}[i]$ be any indecomposable complex that is regular and homogeneous (that is, such that $\tau_{D^b(\text{mod} kQ)} X^\bullet \cong X^\bullet$). Then we claim that $X^\bullet$ belongs to the image $F(\text{mod} A)$ of mod $A$ inside $D^b(\text{mod} kQ)$. This implies the
conclusion, because there exist infinitely many isomorphism classes of regular homogeneous complexes in $R[i]$. By [4] (IV.5.1), p. 175, it suffices to show that $\text{Hom}_{\text{proj}(\text{mod } kQ)}(T^*, X^*[j]) = 0$ for all $j \neq 0$. Now, if $Y^*$ is an indecomposable summand of $T''$, then $\text{Hom}_{\text{proj}(\text{mod } kQ)}(Y^*, Y^*[1]) = 0$ (because $T^*$ is a tilting complex); hence $Y^*$ cannot be homogeneous. This shows that all the indecomposable summands of $T''$ that lie in $R[i]$ belong to tubes distinct from, and hence orthogonal to, the one containing $X^*$. Our claim then follows from the structure of the morphisms in $D^b(\text{mod } kQ)$.

2.4. The following lemma will be useful in the final step of the proof of our main result. We recall that, if $A$ is an algebra, then a module $T_A$ is an almost complete tilting module if there exists an indecomposable module $X$ such that $T \oplus X$ is a tilting module. For the representation theory of tame hereditary algebras, we refer the reader to [6], [9].

**Lemma.** Let $H$ be a tame hereditary algebra, and $T = T' \oplus T''$ be an almost complete tilting module, with $T'$ indecomposable projective and $T''$ regular. Then there exists, up to isomorphism, at most one indecomposable summand $T_0$ of $T''$ on the mouth of each tube such that $\text{Hom}_H(T', T_0) \neq 0$.

**Proof.** By tilting, we may assume that the quiver $Q_H$ of $H$ has the orientation in the Tables of [6]. If the underlying graph of $Q_H$ is $\tilde{A}_n$, then the statement follows from [4] (3.5). The cases where the underlying graph is $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ are easily done by inspection, using the Tables of [6]. We thus only need to consider the case when the underlying graph of $Q_H$ is $\tilde{D}_n$ with the shown orientation.

Since $T$ is almost complete, it has $n$ isomorphism classes of indecomposable summands, of which $n - 1$ are regular. There is therefore one indecomposable summand of $T$ in each of the two tubes of rank 2, and $n - 3$ in the tube of rank $n - 2$. Since the statement is clear for tubes of rank 2 (and even 3), we consider the tube $T$ of rank $n - 2$ and assume $n \geq 6$. By the Tables of [6], the mouth of $T$ contains a sincere indecomposable $M$ with $\text{dim } M = 1 \ldots 11$, and the simple $H$-modules $S(i)$, where $3 \leq i \leq n - 1$. Assume $T' = P(j)$. If $j \in \{1, 2, n, n + 1\}$, then the only module on the mouth of $T$ to which $P(j)$ maps is $M$ and we are done. If not, then $P(j)$ maps exactly to two modules on the mouth of $T$, namely $M$ and $S(j)$. Assume that both $M$ and $S(j)$ are summands of $T''$. Since, for each indecomposable summand $X$ of $T''$, we have $\text{Hom}_H(P(j), \tau X) \cong D \text{Ext}_H^1(X, P(j)) = 0$, the module $\tau X$ is not sincere. Consequently, $X$ lies in the wing $W$ determined by the mouth modules $S(n - 2), S(n - 3), \ldots, S(3), M$. The mesh category of $W$ is isomorphic to the mesh category of the Auslander-Reiten quiver of a hereditary algebra $H'$ of type $\tilde{A}_{n-3}$ with linear orientation. Since $T''$ has $n - 3$ indecomposable summands, it maps under this isomorphism onto a tilting $H'$-module. However, the unique projective-injective $H'$-module must be a summand of any tilting module. This implies that the corresponding module $N$ in $W$ (where $\text{dim } N = 1 \ldots 22 \ldots 21$) is a
summand of $T''$. But $\dim \tau N = 0 \ldots 11$ and, in particular, $\Ext^1_H(N, P(j)) \cong D \Hom_H(P(j), \tau N) \neq 0$, a contradiction.

3. The main result

3.1. We shall use essentially the following lemma and its dual (which we do not state for the sake of brevity). They assert that, under the action of an APR-tilting (or cotilting) module, the images of the points of $\mod A$ under the triangle-equivalence $D^b(\mod A) \cong D^b(\mod kQ)$ get closer together.

Lemma. Let $A$ be an iterated tilted algebra of type $Q$, and let $F : D^b(\mod A) \rightarrow D^b(\mod kQ)$ be a triangle-equivalence. Let $P_A$ be a simple projective module, and $P'_A$ be such that $A_A = P \oplus P'$. Then:

(a) If $\id P = 1$, then $F(\tau^{-1}_A P) \cong \tau^{-1}_{D^b(\mod kQ)} F(P)$.

(b) If $\id P > 1$, then $\Hom_{D^b(\mod kQ)}(F(\tau^{-1}_A P), F(P')) \neq 0$.

Proof. (a) We have an almost split sequence

$$0 \rightarrow P \rightarrow E \rightarrow \tau^{-1}_A P \rightarrow 0$$

where $\pd \tau^{-1}_A P \leq 1$ and $\id P = 1$. By [7] (I.4.7), p.37, we have an almost split triangle in $D^b(\mod A)$

$$P \rightarrow E \rightarrow \tau^{-1}_A P \rightarrow P[1].$$

Hence, applying $F$, we get an almost split triangle in $D^b(\mod kQ)$

$$FP \rightarrow FE \rightarrow F(\tau^{-1}_A P) \rightarrow FP[1].$$

In particular, $F(\tau^{-1}_A P) \cong \tau^{-1}_{D^b(\mod kQ)} F(P)$.

(b) If $\id P \geq 2$, then $\Hom_A(\tau^{-1}_A P, A) \neq 0$. Since $\Hom_A(\tau^{-1}_A P, P) = 0$, then $\Hom_A(\tau^{-1}_A P, P') \neq 0$. Applying $F$ yields $\Hom_{D^b(\mod kQ)}(F(\tau^{-1}_A P), FP') \neq 0$. □

3.2. Lemma. Let $A$ be a representation-finite iterated tilted algebra of euclidean type $Q$ such that any APR-tilting module tilts $A$ to a representation-infinite algebra. Then there exist exactly two consecutive transjective components $\mathcal{C}[t]$ and $\mathcal{C}[t+1]$ that contain summands of the standard complex, and furthermore, $\bigcup_{j < t} \mathcal{R}[j]$ contains no summand of the standard complex.

Proof. By (2.3), there exist $t \neq s$ in $\mathbb{Z}$ such that $\mathcal{C}[t]$ and $\mathcal{C}[s]$ contain summands of the standard complex. We may assume that $t < s$ and that $\mathcal{C}[r]$ contains no summand of the standard complex if $r < t$ or $r > s$.

Assume that there exists $a \in (Q_A)_0$ such that $T^\bullet(a) \in \bigcup_{j < t} \mathcal{R}[j]$. Then there exist a sink $a \in (Q_A)_0$ and an integer $j < t$ such that $T^\bullet(a) \in \mathcal{R}[j]$. The endomorphism algebra of the APR-tilting module at $a$ is clearly iterated tilted. It is moreover representation-finite, by (2.3), since $\mathcal{C}[t] \neq \mathcal{C}[s]$ contain summands of its standard complex. This contradicts our hypothesis on $A$, and shows that $\bigcup_{j < t} \mathcal{R}[j]$ contains no summand of the standard complex.

We claim that $s = t+1$. If this is not the case, then $s > t + 1$ and, by hypothesis, there exists a sink $a \in (Q_A)_0$ such that $T^\bullet(a) \in \mathcal{C}[t]$. Then $T^\bullet(a)$ is the unique indecomposable summand of $T^\bullet = F(A_A)$ to lie in $\mathcal{C}[t]$ (otherwise, considering the APR-tilting module at $a$ yields a contradiction, as above). On the other hand,
there is no indecomposable summand of $T^\bullet$ in $\mathcal{C}[t + 1]$ (otherwise, consider again the APR-tilting module at $a$). We shall show that $F(\tau^{-1}_A P(a)) \in \mathcal{C}[t + 1]$. This will imply the statement since, considering the endomorphism algebra of the APR-tilting module at $a$, there exist summands of its standard complex in $\mathcal{C}[t + 1]$ and $\mathcal{C}[s]$, again a contradiction.

The almost split sequence $0 \to P(a) \to E \to \tau^{-1}_A P(a) \to 0$ induces a (not necessarily almost split) triangle in $D^b(\text{mod } A)$

$$P(a) \xrightarrow{u} E \xrightarrow{v} \tau^{-1}_A P(a) \xrightarrow{w} P(a)[1]$$

such that $u \neq 0$, $v \neq 0$ and $w \neq 0$. Applying $F$ yields a triangle in $D^b(\text{mod } kQ)$

$$FP(a) \xrightarrow{Fu} FE \xrightarrow{Fv} F(\tau^{-1}_A P(a)) \xrightarrow{Fw} FP(a)[1].$$

Since $Fu \neq 0$ and $E_A$ is projective, the considerations above show that $FE \in R[t]$. Taking a complete slice in $\mathcal{C}[t]$ having $T^\bullet(a) = FP(a)$ as a unique source, we get a hereditary algebra $H$ such that we have a triangle-equivalence $D^b(\text{mod } A) \cong D^b(\text{mod } H)$ under which $FP(a)$ maps onto the unique simple projective $H$-module. On the other hand, $FE$ maps into the regular component of $\text{mod } H$. Since we can assume that $H = kQ$, we can (and will) identify $FP(a)$, $FE$, etc. to their images. Since $FP(a)$ is simple, $Fu$ is a monomorphism. Hence we have a diagram

$$
\begin{array}{ccc}
FP(a) & \xrightarrow{Fu} & FE \\
| & & | \\
FP(a) & \xrightarrow{Fv} & F(\tau^{-1}_A P(a)) \\
| & & | \\
FP(a) & \xrightarrow{Fw} & FP(a)[1]
\end{array}
\quad
\begin{array}{ccc}
FP(a) & \xrightarrow{Fu} & FE \\
| & & | \\
FP(a) & \xrightarrow{Fv} & \text{Coker } Fu \\
| & & | \\
FP(a) & \xrightarrow{Fw} & FP(a)[1]
\end{array}
$$

where the dotted map exists and is an isomorphism by the elementary properties of triangulated categories [2] (I.1.1), p.2. Thus $\text{Coker } Fu \cong F(\tau^{-1}_A P(a))$. Now $\text{Coker } Fu$ is not regular, since, otherwise, the regularity of $FE$ implies that $FP(a)$ is also regular (because the regular $H$-modules form an abelian subcategory of $\text{mod } H$). Since $\text{Hom}_{D^b(\text{mod } H)}(\text{Coker } Fu, FP(a)[1]) \neq 0$, we deduce that $\text{Coker } Fu \in \mathcal{C}[t + 1]$. \hfill \Box

Proof of the theorem. Let $A$ be a representation-finite iterated tilted algebra of euclidean type $Q$. By applying a finite sequence of APR-tilting modules, we obtain, by [7] (IV.7.4), p. 190, an iterated tilted algebra $B$ of type $Q$ having the following property: $B$ is representation-finite, but the endomorphism algebra of any APR-tilting $B$-module is representation-infinite. Clearly, we may assume that $A = B$.

By (2.3), there exist $t \neq s$ in $\mathbb{Z}$ such that $\mathcal{C}[t]$ and $\mathcal{C}[s]$ contain summands of the standard complex $T^\bullet = FA_1$, where $F$ denotes as usual a triangle equivalence $D^b(\text{mod } A) \to D^b(\text{mod } kQ)$. We may assume that $0 = t < s$ and that $\mathcal{C}[r]$ contains no summand of $T^\bullet$ for $r < 0$ and $r > s$. By the dual of (3.2), we have further that $s = 1$ and $\bigcup_{j \geq 1} \mathcal{R}[j]$ contains no summand of the standard complex.

We thus know that $\mathcal{C}[0]$ and $\mathcal{C}[1]$ contain indecomposable summands of $T^\bullet$ and that $\bigcup_{j \leq 0} \mathcal{R}[j]$ may. We now show that we may assume that no summand of $T^\bullet$
lies in $\bigcup_{j<0} R[j]$. If this is the case, then there exist a sink $a \in (Q_A)_0$ and $j < 0$ such that $T^\bullet(a) = FP(a) \in R[j]$. Let $b \in (Q_A)_0$ be such that $T^\bullet(b) = FP(b) \in C[1]$. The endomorphism algebra of the APR-tilting module at $a$ is still representation-finite (by (2.3)) and there exists a path from $F(\tau_A^{-1}P(a))$ to $FP(b)$. Applying successively a sequence of APR-tilting modules corresponding to those summands of $T^\bullet$ lying in $\bigcup_{j<0} R[j]$, and using (3.1), we thus reach an algebra $C$ having the following properties:

(i) $C$ is representation-finite, and
(ii) $FC_C \in C[0] \cup R[0] \cup C[1]$, where $F$ denotes, as usual, a triangle-equivalence $F : D^b(\text{mod } C) \to D^b(\text{mod } kQ)$.

We now claim that $C$ can be reduced (by applying a finite sequence of APR-tilting or APR-cotilting modules) to either a representation-finite tilted algebra – and then we are done – or else to the case when the standard complex $FC_C = T^\bullet$ has a unique indecomposable summand $T^\bullet_0$ in $C[0]$ and a unique one $T^\bullet_1$ in $C[1]$.

Assume that the latter situation does not occur. In particular, $T^\bullet = FC_C$ has more than one indecomposable summand in $C[0]$ or $C[1]$, say the former. Let $T^\bullet_0$ denote a “last” indecomposable summand of $T^\bullet$ to lie in $C[0]$, that is, let $T^\bullet_0$ be such that there exists no path in $C[0]$ from $T^\bullet_0$ to another indecomposable summand of $T^\bullet$. Taking a complete slice in $C[0]$ having $T^\bullet_0$ as unique source yields a hereditary algebra $H$ such that $D^b(\text{mod } C) \cong D^b(\text{mod } H)$. We may assume that the sequence of APR-tilting and cotilting modules used to reach the algebra $C$ has the property that the number of isomorphism classes of indecomposable summands of $T^\bullet$ lying outside mod $H$ is the least possible. It follows from (3.1) that there exists a finite sequence of APR-tilting modules transforming $C$ to $D$ (say) such that there exists a sink $a \in (Q_D)_0$ with $T^\bullet(a)$ not in mod $H$, but the middle term $E$ of the almost split sequence

$$0 \to P(a) \to E \to \tau_D^{-1}P(a) \to 0$$

has none of its indecomposable summands lying in mod $H$. By an argument similar to the one used in (3.2), we deduce that $F(\tau_D^{-1}P(a))$ lies in mod $H$, and this contradicts the assumed minimality. This completes the proof of our claim.

We note that, if all the indecomposable summands of $T^\bullet = FC_C$ lie inside mod $H$, then the tilting complex $T^\bullet$ is a tilting module and $C$ is a tilted algebra.

We now consider the case where $FC_C$ has a unique indecomposable summand $T^\bullet_0$ in $C[0]$ and a unique one $T^\bullet_1$ in $C[1]$. Clearly, this implies that $T^\bullet = FP(a)$, with $a$ a sink in $Q_C$ and $T^\bullet_1 = FP(b)$, with $b$ a source.

We claim that we may assume $a$ to be the only sink, and $b$ the only source in $Q_C$. Indeed, if $a' \neq a$ is another sink in $Q_C$, then $T^\bullet(a')$ necessarily lies in $R[0]$. We claim that $F\tau_C^{-1}P(a') \in R[0]$. Indeed, by (3.1), either $P(a')_C = 1$, and then $F\tau_C^{-1}P(a') = \tau_D^{-1}P(c) \in R[0]$, or else $P(a')_C > 1$, and then $\text{Hom}_{D^b(\text{mod } H)}(F\tau_C^{-1}P(a'), FP') \neq 0$, where $FP' = \bigoplus_{c \neq a'} P(c)$. Consequently, there exists a path $F\tau_C^{-1}P(a') \to \cdots \to T^\bullet_1$ in $D^b(\text{mod } H)$. Therefore the existence of a morphism $FE \to F\tau_C^{-1}P(a')$ implies that $T^\bullet_1$ cannot be a summand of $FE$. Thus $FE \in R[0]$. Since $R[0]$ is abelian, we deduce, as in (3.2), that $F\tau_C^{-1}P(a') \in R[0]$, as required.
We now consider the regular summands of $T^\bullet$, lying in $R[0]$. We recall that the endomorphism algebra of a partial tilting module lying in a tube of a tame hereditary algebra is a direct product of tilted algebras of type $A_n$ (see, for instance, [4] (1.4)). This first implies that the summands of $T^\bullet$ belonging to a given tube may be assumed to lie on a sectional path: indeed, if this is not the case, then it follows from (2.4) that $Q_C$ would have a source or a sink distinct from $a$ or $b$, a contradiction. Next, those summands of $T^\bullet$ belonging to a given tube are equal to all exceptional points on the sectional path on which they lie: this is because a euclidean diagram has $n + 1$ points; hence $T^\bullet = FC$ has $(n + 1)^2 = n$ summands in $R[0]$, and an indecomposable summand of $T^\bullet$ is the image of an exceptional $H$-module under the embedding of mod $H$ into $D^b$(mod $H$). This argument also shows that the endomorphism algebra of the direct sum of those summands of $T^\bullet$ that lie in $R[0]$ is the direct product of (at most three) hereditary Nakayama algebras $H_i$ of quiver

\[
\begin{array}{cccccccccc}
\bullet & \longleftrightarrow & \bullet & \longleftrightarrow & \cdots & \longleftrightarrow & \bullet & \longleftrightarrow & \bullet
\end{array}
\]

Now, since $a$ is the only sink and $b$ the only source in $Q_C$, there must exist, for each $H_i$, an arrow from the unique sink of $Q_{H_i}$ to $a$, and an arrow from $b$ to the unique source of $Q_{H_i}$. Thus $C = \text{End } T^\bullet$ has for quiver

\[
\begin{array}{cccccccccc}
\bullet & \longleftrightarrow & \bullet & \longleftrightarrow & \cdots & \longleftrightarrow & \bullet & \longleftrightarrow & \bullet
\end{array}
\quad \text{or} \quad
\begin{array}{cccccccccc}
\bullet & \longleftrightarrow & \bullet & \longleftrightarrow & \cdots & \longleftrightarrow & \bullet & \longleftrightarrow & \bullet
\end{array}
\]

We now determine the possible relations on this quiver. Assume first that there exists a non-zero morphism from $T^\bullet(a)$ to $T^\bullet(b)$. There exists a hereditary algebra $H'$ having a unique simple projective module and a triangle-equivalence $D^b$(mod $H'$) $\cong D^b$(mod $H$) such that $T^\bullet(a)$ maps onto the unique simple projective $H'$-module. Since there exists a non-zero morphism from $T^\bullet(a)$ to $T^\bullet(b)$, it follows that $T^\bullet(b)$ maps onto an indecomposable $H'$-module. But then $T^\bullet$ maps onto a tilting $H'$-module, and consequently $C = \text{End } T^\bullet$ is a tilted algebra. If, on the other hand, there exists no non-zero morphism from $T^\bullet(a)$ to $T^\bullet(b)$, then, since $C$ is a representation-finite iterated tilted algebra, it is representation-directed (by [7], (IV.3.6), p. 169); hence there exists a zero-relation on each path from $b$ to $a$ in the quiver of $C$. Now, it follows from the structure of the morphisms in the derived category that a relation can only go from the point $b$ to the point $a$. Therefore, $C$ is a one-point extension of a hereditary algebra by a direct sum of (two or three) indecomposable injective modules; hence it is tilted. This completes the proof of our theorem. (We observe that in the last step of the proof, since $C$ is not simply connected, it is in fact iterated tilted of type $A_n$, and it follows easily from the classification in [3] that $C$ is a one-point extension of a hereditary algebra of type $A_3$ by the direct sum of two simple injective modules).
3.4. **Examples.** (a) Our first example shows that Roldán’s conjecture, in its original form, is not true. Let $A$ be the algebra given by the quiver

![Quiver Diagram](image)

bound by $\beta \gamma = 0$, $\delta \varepsilon = 0$, $\alpha \delta = 0$. Then $A$ is a representation-finite iterated tilted algebra of type $A_4$ (see [3]), it is clearly not tilted. It is easily verified that $A$ has exactly four non-projective and multiplicity-free tilting modules, namely

(i) $T_A = \mathsf{2}^{3} \mathsf{1}^{2} \mathsf{3}^{1} \mathsf{2}^{3} \mathsf{4}^{5} \mathsf{2}$;

(ii) $T_A = \mathsf{2}^{1} \mathsf{3}^{3} \mathsf{1}^{3} \mathsf{3}^{2} \mathsf{4}^{5} \mathsf{2}$;

(iii) $T_A = \mathsf{2}^{1} \mathsf{3}^{3} \mathsf{2}^{3} \mathsf{2}^{3} \mathsf{4}^{5} \mathsf{2}$;

(iv) $T_A = \mathsf{2}^{1} \mathsf{3}^{3} \mathsf{2}^{3} \mathsf{3}^{3} \mathsf{4}^{5} \mathsf{2}$.

In each case, $\text{End } T_A$ is representation-infinite (because it contains a representation-infinite hereditary algebra as a full convex subcategory).

(b) Our second example shows that our theorem is not valid for iterated tilted algebras of wild type. Let $B$ be the wild hereditary algebra given by the quiver

![Quiver Diagram](image)

and $C = B/\text{rad}^2 B$. Then it is easily verified that $C$ is a representation-finite iterated tilted algebra of type $Q_B$, and that any going-up tilting series from $C$ to $B$ passes through a non-hereditary representation-infinite algebra.

(c) It is reasonable to ask whether the procedure of the theorem can be achieved by performing first all the APR-tilts, then all the APR- cotilts. This is not possible as is shown by the case of the algebra $D$, which is the one-point coextension of the algebra $A$ of example (a) above by the simple module $S(1)$. 

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