

FINITE GENERATION PROPERTIES FOR FUCHSIAN GROUP VON NEUMANN ALGEBRAS TENSOR $B(H)$

FLORIN RĂDULESCU

(Communicated by David R. Larson)

ABSTRACT. We prove that the algebra $\mathcal{A} = \mathcal{L}(F_N) \otimes B(H)$, F_N a free group with finitely many generators, contains a subnormal operator J such that the linear span of the set $\{(J^*)^n J^m | n, m = 0, 1, 2, \dots\}$ is weakly dense in \mathcal{A} . This is the analogue for the II_∞ factor $\mathcal{L}(F_N) \otimes B(H)$, N finite, of a well known fact about the unilateral shift S on a Hilbert space K : the linear span of all the monomials $(S^*)^n S^m$ is weakly dense in $B(K)$.

We also show that for a suitable space H^2 of square summable analytic functions, if P is the projection from the Hilbert space L^2 of all square summable functions onto H^2 and $M_{\bar{j}}$ is the unbounded operator of multiplication by \bar{j} on L^2 , then the (unbounded) operator $PM_{\bar{j}}(I - P)$ is nonzero (with nonzero domain).

In this paper we prove the existence of finite systems \mathcal{G} of generators, with special properties, for the von Neumann algebras $\mathcal{L}(\Gamma) \overline{\otimes} B(H)$, obtained by tensoring the group von Neumann algebra of a fuchsian group Γ with the bounded linear operators on an infinite dimensional, separable, Hilbert space H . The results show that the linear span of ordered monomials that are products of powers of adjoints of elements in \mathcal{G} with powers of elements in \mathcal{G} is weakly dense in the algebra. In addition the systems of generators are a commuting family, consisting of subnormal operators.

If $\Gamma = PSL(2, \mathbb{Z})$, then there exists a single subnormal operator J , in the algebra $\mathcal{A} = \mathcal{L}(PSL(2, \mathbb{Z})) \otimes B(H)$ (which by the results in ([Dyk], [Ra2]) is isomorphic to $\mathcal{L}(F_N) \otimes B(H)$, F_N a free group with finitely many generators), such that linear span of the set $\{(J^*)^n J^m | n, m = 0, 1, 2, \dots\}$ is weakly dense in \mathcal{A} . This is the analogue for the II_∞ factor $\mathcal{L}(F_N) \otimes B(H)$, N finite, of a well known fact about the unilateral shift S on a Hilbert space K : the linear span of all the monomials $(S^*)^n S^m$ is weakly dense in $B(K)$. The operator J comes from a Toeplitz operator having as symbol Dedekind's modular invariant function j and acting on a suitable Hilbert space of analytic functions.

The construction in this paper may also be used to provide examples for Toeplitz operators with unbounded symbol, with unexpected behaviour, and non-closability for multiplication operators, with bounded symbol, in Sobolev type Hilbert spaces

Received by the editors September 22, 1998.

1991 *Mathematics Subject Classification*. Primary 46L35; Secondary 46L37, 46L57, 81S99, 11F99.

The author's research was supported in part by the grant DMS 9622911 from the National Science Foundation. The author is a member of the Institute of Mathematics, Romanian Academy, Bucharest.

associated to differential operators commuting with the laplacian. More precisely we will show that for a suitable space H^2 of square summable analytic functions, if P is the projection from the Hilbert space L^2 of all square summable functions onto H^2 and $M_{\bar{j}}$ is the unbounded operator of multiplication by \bar{j} on L^2 , then the (unbounded) operator $PM_{\bar{j}}(I - P)$ is nonzero (with nonzero domain). This is in contrast to what happens when ϕ is a bounded analytic function in H^2 ; then $PM_{\bar{\phi}}(I - P) = 0$.

In proving this result we will rely on the construction outlined in [Ra3] on the equivariant Berezin quantization of the upper half-plane and on a remarkable result of Murray and von Neumann about the algebra of unbounded operators affiliated to a type II_1 factor ([MvN]).

1. TOEPLITZ OPERATORS WITH UNBOUNDED SYMBOL

In this section we outline the construction of the Toeplitz operator with unbounded and antianalytic symbol the modular function j . This operator is not closable and this will allow us to construct the non zero, unbounded operator of the form $PM_{\bar{j}}(I - P)$.

Recall some of the notations that were used in [Ra3]. Let H_t be the Hilbert space of square summable, analytic functions on \mathbb{H} , with respect to the measure $d\nu_t = (\text{Im } z)^{t-2}d\bar{z}dz$. In [Pu], [Sa] it was proven that there exists a one parameter family $(\pi_t)_{t>1}$, of irreducible, projective unitary representation of $PSL(2, \mathbb{R})$ on H_t that extends the analytic discrete series of representations for $PSL(2, \mathbb{R})$. Moreover the methods in [GHJ], along with the trace formula in [Pu], were used in ([Ra3]) to show that for any fuchsian group Γ , the von Neumann algebra $\{\pi_t(\Gamma)\}''$ is a type II factor acting on H_t . The Murray–von Neumann dimension for this algebra acting on H_t is proportional to t and the covolume of Γ . In fact, the commutant algebras $\mathcal{A}_t = \{\pi_t(\Gamma)\}''$ are isomorphic to the twisted group von Neumann algebra $\mathcal{L}(\Gamma, \sigma^t)_{(t-1)/\text{covolume } \Gamma}$ (or to $\mathcal{L}(\Gamma, \sigma^t) \otimes B(H)$ if the covolume of Γ is infinite), with σ^t the cocycle corresponding to the projective representation π_t .

As shown in [GHJ], automorphic forms g (for Γ) of weight k , where k is a strictly positive integer, correspond to bounded intertwining operators S_g^t from H_t into H_{t+k} . S_g^t is the multiplication operator with g acting on H_t . The intertwining property means that

$$\pi_{t+k}(\gamma)S_g^t = S_g^t\pi_t(\gamma), \gamma \in \Gamma.$$

Let f be any measurable, bounded Γ -invariant function on \mathbb{H} . Let M_f^t be the multiplication operator by f on H_t and let P_t be the projection operator from bounded square summable functions on \mathbb{H} onto H_t . The Toeplitz operator $T_f^t = P_tM_f^tP_t$ clearly commutes with $\{\pi_t(\Gamma)\}$ and hence T_f^t belongs to \mathcal{A}_t .

Also, we consider in this paper Toeplitz operators, not necessarily bounded, having as symbols Γ invariant, analytic functions on \mathbb{H} (i.e. automorphic forms of order 0). Unfortunately, as we show below, though affiliated with the commutant algebras, such an operator is not densely defined, nor is it closable. One possible symbol function, for $\Gamma = PSL(2, \mathbb{Z})$, is the classical modular invariant function $j = \frac{\Delta}{G_2^3}$.

Lemma 1. *Let Γ be a fuchsian group, of finite covolume, and let j be an analytic, Γ -invariant function on \mathbb{H} . Assume that $j = \frac{a}{b}$ is the quotient of two automorphic forms for Γ having the same weight m . Assume that the functions*

$z \rightarrow |a(z)|^2(\text{Im } z)^{m-2}$, $z \rightarrow |b(z)|^2(\text{Im } z)^{m-2}$, are bounded. Also we assume that a, b have not all their zeroes and poles in common.

For a bounded, measurable, Γ invariant function f on \mathbb{H} , let T_f^t be the Toeplitz operator on H_t with symbol f . Clearly, T_f^t belongs to the commutant $\mathcal{A}_t = \{\pi_t(PSL(2, \mathbb{Z}))'\}$ and the set of all such operators is a weakly dense subspace of the Hilbert space $L^2(\mathcal{A}_t, \tau)$, associated with the trace τ on \mathcal{A}_t (see [Ra1]).

Then the operator $T_f^t \rightarrow T_{\bar{j}f}^t$, whose domain is a dense subset of $L^2(\mathcal{A}_t, \tau)$, is non closable, for any $t > 1$.

Moreover the same holds true if we replace \bar{j} by $|j|^2$ or by $\frac{\bar{j}}{j}$.

Proof. The hypothesis shows that the bounded (see [GHJ]) operators S_a^t, S_b^t on H_t into H_{t+k} , defined by by multiplication with a and respectively b , have nonequal projections onto the closure of their ranges. Let c be any other automorphic form Γ having the same order as a, b . Let Ψ be the operator in the statement. Then for any Γ -invariant, bounded, measurable function h we will have that

$$\Psi((S_a^t)^*T_h^{t+k}S_c^t) = (S_b^t)^*T_h^{t+k}S_c^t.$$

This holds because obviously $(S_a^t)^*T_h^{t+k}S_c^t = T_{\bar{a}hc}^t$ and $(S_b^t)^*T_h^{t+k}S_c^t = T_{\bar{b}hc}^t$.

Let e_a be the projections onto the range of S_a^t , which is a subspace of H_{t+k} . As the set of all T_h^{t+k} , when h runs through the bounded, measurable, Γ -equivariant functions on \mathbb{H} , is weakly dense in the commutant \mathcal{A}_{t+k} , it follows that for any x in \mathcal{A}_{t+k} we may find a sequence of such functions h_n on \mathbb{H} such that $T_{h_n}^{t+k}$ converges weakly to $(1 - e_a)x$ and hence $(S_a^t)^*T_{h_n}^{t+k}S_c^t$ converges weakly to 0. Then $(S_b^t)^*T_{h_n}^{t+k}S_c^t$ converges weakly to $(S_b^t)^*(1 - e_a)xS_c^t$. If Ψ were closable, then it would follow that $(S_b^t)^*(1 - e_a)xS_c^t$ is zero for all x in \mathcal{A}_t . Since x is arbitrary, it follows that $(S_b^t)^*(1 - e_a) = 0$.

Because the operators S_a^t, S_b^t have non equal projections onto the closure of their ranges, while those projections have the same trace (as both operators are injective), this is impossible. This completes the proof for \bar{j} .

The statement for the other two functions is proved in a similar way: multiplication by $|j|^2$ maps $(S_a^t)^*T_h^{t+k}S_a^t$ into $(S_b^t)^*T_h^{t+k}S_b^t$ (and this is a positive map) and multiplication by $\frac{\bar{j}}{j}$ maps $(S_a^t)^*T_h^{t+k}S_b^t$ into $(S_b^t)^*T_h^{t+k}S_a^t$. □

Remark 2. With the notations in the above statement let F be a fundamental domain for the action of Γ on \mathbb{H} . Let $B_t = B_t(\Delta)$ be the Berezin operator defined by

$$B_t f(z) = \int_{\mathbb{H}} f(w) \left[\frac{(\text{Im } z)(\text{Im } w)}{\bar{z} - w} \right]^{t-2} (\text{Im } w)^{-2} d\bar{w}dw, \quad z \in \mathbb{H}.$$

Note that the operator B_t commutes with the invariant laplacian ([Be]) on $L^2(F, (\text{Im } w)^{-2}d\bar{w}dw)$. Then $L^2(\mathcal{A}_t, \tau)$ is identified with the the completion of a dense subset of $L^2(F, (\text{Im } w)^{-2}d\bar{w}dw)$ with the scalar product

$$\langle f, g \rangle = (B_t(\Delta)f, g)_{L^2(F)}.$$

With this scalar product, the operator of multiplication by \bar{j} on a dense subspace of $L^2(F, (\text{Im } w)^{-2}d\bar{w}dw)$ is non closable. Moreover the same holds true if we replace \bar{j} by $|j|^2$ or by $\frac{\bar{j}}{j}$.

The following corollary shows that one of the properties that is valid for Toeplitz operators with bounded, antianalytic symbol fails for unbounded symbols (see also [JS], [Saf]).

Corollary 3. *Let Γ be a fuchsian group of finite covolume, and let $j = \frac{a}{b}$ be an analytic, Γ invariant function on \mathbb{H} as above. Let P_t be the projection from $L^2(\mathbb{H}, (\text{Im } z)^{t-2} d\bar{z}dz)$ onto $H_t = H^2(\mathbb{H}, (\text{Im } z)^{t-2} d\bar{z}dz)$. Let $M_{\bar{j}}$ be the multiplication operator with \bar{j} which is defined on a dense subset of the Hilbert space $L^2(\mathbb{H}, (\text{Im } z)^{t-2} d\bar{z}dz)$.*

Then $P_t M_{\bar{j}}(1 - P_t)$ is nonzero (in particular has nonzero domain).

Proof. Assume the contrary. Let h, h_1 be any Γ -invariant, bounded, measurable, real valued functions h, h_1 on \mathbb{H} , such that the operators $T_h^t, T_{h_1}^t$ are injective. Also, assume that the supports of h, h_1 are so that the functions $\bar{j}h, \bar{j}h_1$ are bounded.

Let Z_h, Z_{h_1} be the closable operators ([MvN]), defined by

$$Z_h = T_{\bar{j}h}^t (T_h^t)^{-1}, Z_{h_1} = T_{\bar{j}h_1}^t (T_{h_1}^t)^{-1}.$$

Our assumption implies that

$$T_{\bar{j}h}^t \zeta = T_{\bar{j}h_1}^t \zeta_1 \text{ whenever } T_h^t \zeta = T_{h_1}^t \zeta_1.$$

This implies that the closable, unbounded operators Z_h, Z_{h_1} coincide on a densely defined core and hence that they are equal (again by [MvN]).

Hence there exists a unique, closed operator, affiliated with $\mathcal{A}_t = \{\pi_t(\Gamma)\}'$ such that for any real valued, bounded, measurable functions h , with T_h^t invertible, we have (by [MvN])

$$Z T_h^t = T_{\bar{j}h}^t.$$

But this is impossible by the previous statement. □

Corollary 4. *With the notations in the previous statement, the same conclusion holds if j is replaced by $h \circ j$, where h is any univalent entire function h .*

Proof. By examining the above argument, we see that in fact we proved that if K is any compact subset of the interior of F such that j is bounded when restricted to K and χ_K is the characteristic function of K , then

$$P_t M_{\bar{j}}[(\text{Id} - P_t) \wedge \chi_K] \neq 0.$$

Now assume that the (bounded) linear operator $M_{\overline{f \circ j}}$ would have the property that

$$M_{\overline{f \circ j}}[(\text{Id} - P_t) \wedge \chi_K](H_t) \subseteq (\text{Id} - P_t)(H_t).$$

It would then follow that $M_{\overline{f \circ j}}$ would also have the property that

$$M_{\overline{f \circ j}}[(\text{Id} - P_t) \wedge \chi_K](H_t) \subseteq ((\text{Id} - P_t) \wedge \chi_K)(H_t).$$

Since f is univalent, the same would then hold true about j instead of $f \circ j$, and this we know to be false. □

Questions. (i). *Is the above statement true if one drops the univalence condition on f ?*

(ii). *If $t > 24$, $\Gamma = PSL(2, \mathbb{Z})$ and $j = \frac{\Delta}{G_3^2} = \frac{\Delta^2}{G_4^3 \Delta}$, then it is clear that the domain of $M_{\bar{j}}$ intersects H_t nontrivially. Does this hold for smaller values of t ?*

2. GENERATION PROPERTIES

In this section we construct the system of generators for algebras of the form $\mathcal{L}(\Gamma) \otimes B(H)$ where Γ is a fuchsian group with special properties. We will then specialize to the case $\Gamma = PSL(2, \mathbb{Z})$ and show one can reduce the system of generators to a single generator in this case.

Assume first that Γ is a fuchsian group (of infinite covolume). Assume that $H^\infty(\mathbb{H}/\Gamma)$ has a sufficiently rich structure so that there are a finite number of bounded analytic functions that separate points on \mathbb{H}/Γ . In this case, by using the methods developed in [Ra3] we can show that $\mathcal{L}(\Gamma) \otimes B(K)$ has a set of generators with the properties in the following proposition.

Proposition 5. *Let Γ be a fuchsian group such that $H^\infty(\mathbb{H}/\Gamma)$ contains functions h_1, \dots, h_k that separate the points on \mathbb{H} . (For examples of such groups see [St].) Let K be an infinite dimensional Hilbert space. Then there exist commuting, bounded subnormal operators Z_1, \dots, Z_k in $\mathcal{B} = \mathcal{L}(\Gamma) \otimes B(K)$ such that*

$$\mathcal{B} = \overline{\text{Sp} \{ (Z_i^n)^* Z_j^m | n, m \in \mathbb{N}, i, j = 1, 2, \dots, k \} }^{\text{weak}}.$$

Proof. We consider the algebra $\mathcal{B} = \{ \pi_t(\Gamma) \}''$. Since Γ has infinite covolume, it follows that \mathcal{B} is isomorphic to $\mathcal{L}(\Gamma) \otimes B(K)$. Let $Z_i = T_{h_i}^t, i = 1, \dots, k$. Let c be an arbitrary element in $\mathcal{B} \cap L^1(\mathcal{B}, \tau)$, with zero kernel. Assume that $a \in \mathcal{B}$ is orthogonal to the following subspace of $L^1(\mathcal{B}, \tau)$:

$$\overline{\text{Sp} \{ c(Z_i^n)^* Z_j^m | n, m \in \mathbb{N}, i = 1, \dots, k \} }^{\text{weak}}.$$

We use the notations in [Ra3]. Let $(\hat{ac})(\bar{z}, z), z \in \mathbb{H}$, be the Berezin symbol of $ac \in \mathcal{B} \cap L^1(\mathcal{B}, \tau)$. The trace formula in [Ra3] shows that

$$\int_F (\hat{ac})(\bar{z}, z) \overline{h_i^n(z)} h_j(z) (\text{Im } z)^{t-2} d\bar{z} dz = 0.$$

Hence $ac = 0$ and hence $a = 0$. Consequently

$$\text{Sp} \{ c(Z_i^n)^* Z_j^m | n, m \in \mathbb{N}, i = 1, \dots, k \}$$

is a weakly dense subspace of $L^1(\mathcal{B}, \tau)$ and consequently, since c has zero kernel, it follows that

$$\overline{\text{Sp} \{ (Z_i^n)^* Z_j^m | n, m \in \mathbb{N}, i, j = 1, \dots, k \} }^{\text{weak}} = \mathcal{B}.$$

□

Proposition 6. *Let E be an open, Γ -invariant, subset of \mathbb{H} , such that j is bounded on E . Let $H_t(E)$ be the subspace of $L^2(\mathbb{H}, \nu_t)$, consisting of square summable, analytic functions on E (that are extended with 0 outside E).*

Then $H_t(E)$ is Γ -invariant, with respect to the unitary representation π_t of $PSL(2, \mathbb{R})$ (restricted to Γ), on $L^2(\mathbb{H}, \nu_t)$. Moreover, $H_t(E)$ is an infinite, Hilbert, left module over $\mathcal{L}(\Gamma)$ (and a left submodule, over $\mathcal{L}(\Gamma)$, of $L^2(\mathbb{H}, \nu_t)$).

Let Z be the Toeplitz operator on $H_t(E)$ with symbol $j|_E$. Then the commutant of $\pi_t(\Gamma)$ in $B(H_t(E))$ is the weak closure of the linear span of the set $\{ (Z^)^n Z^m | n, m = 0, 1, 2, \dots \}$.*

Proof. Clearly $H_t(E)$ is a submodule of $L^2(\mathbb{H}, \nu_t)$ over $\mathcal{L}(\Gamma)$. Let $k_E = k_E^t$ be the reproducing kernel for $H_t(E)$ and let $F(E)$ be a fundamental domain for Γ acting on E . Then the Murray-von Neumann dimension of $H_t(E)$ as a left module over

$\mathcal{L}(\Gamma)$ is equal to $\int_{F(E)} k_E(z, z) d\nu_t(z)$. This is obviously infinite, as no subnormal operator (that is not normal) can live in a type II_1 factor. (I owe this last remark to Ken Dykema.)

Moreover, as in [Be], if A is an operator acting on $H_t(E)$, we define its Berezin kernel (with respect $H_t(E)$) to be

$$k_A^E(z, \zeta) = \frac{\langle Ae_z^{E,t}, e_\zeta^{E,t} \rangle}{\langle e_z^{E,t}, e_\zeta^{E,t} \rangle}, z, \zeta \in E,$$

where $e_z^{E,t}$, for z in E , is the evaluation vector in $H_t(E)$ at z .

If A is an operator in $B(H_t(E))$ that commutes with the action of Γ , and if A is in the trace ideal of the commutant algebra (which is a type II_∞ factor), then the trace of A in the commutant algebra is given (as in [Ra3]) by the formula

$$\int_{F(E)} k_A^E(z, z) \frac{k_E^t(z, z)}{(\operatorname{Im} z)^t} d\nu_0(z).$$

Note that the factor $\frac{k_E^t(z, z)}{(\operatorname{Im} z)^t}$ is Γ -invariant. This is the trace on the commutant of Γ on $L^2(\mathbb{H}, \nu_t)$ that is normalized by giving value $\frac{t-1}{\pi}$ to $H^2(\mathbb{H}, \nu_t)$.

Let f be a bounded Γ -invariant function on E and let $T_{f,E}^t$ be the Toeplitz operator on $H_t(E)$ with symbol f . As in [Ra3] (see also [Be]), the formula for the trace in the commutant algebra of $AT_{f,E}^t$ is

$$\int_{F(E)} k_A^E(z, z) f(z) \frac{k_E^t(z, z)}{(\operatorname{Im} z)^t} d\nu_0(z).$$

Hence, if an operator A , commuting with Γ and in the trace ideal of the commutant algebra, is orthogonal (with respect to the trace on the commutant) on all Toeplitz operators on $H_t(E)$, with Γ -equivariant symbol, then it follows that $A = 0$. Hence the linear span of all Toeplitz operators, with bounded, Γ -equivariant symbols, is weakly dense in the commutant. Since the symbol of $(Z^*)^n Z^m$ is $(\bar{j}|_E)^n (j|_E)^m$, the statement follows by the Stone-Weierstrass theorem. \square

As a corollary of the previous proposition we get our main result

Theorem 7. *In the algebra $\mathcal{A} = \mathcal{L}(PSL(2, \mathbb{Z}) \otimes B(H)) = \mathcal{L}(F_N) \otimes B(H)$, where N is finite, there exists a bounded, subnormal operator Z , such that \mathcal{A} is the weak closure of linear span of the set $\{(Z^*)^n Z^m | n, m = 0, 1, 2, \dots\}$.*

ACKNOWLEDGEMENT

The author is indebted to K. Dykema and P. de la Harpe for very useful comments.

REFERENCES

- [Be] F. A. Berezin, Quantization in complex symmetric spaces, *Math USSR Izvestija*, **9**(1975), 341-379. MR **58**:22691
- [Co] J. B. Conway, *Subnormal Operators*, Research Notes in Mathematics v. 51, Pitman Advanced Publ. Prog., Boston, 1982. MR **83i**:47030
- [Dyk] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, *Duke Math. J.* **69**, 97-119 (1993). MR **93m**:46071
- [GHJ] F. Goodman, P. de la Harpe, V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer Verlag, New York, Berlin, Heidelberg, 1989. MR **91c**:46082

- [JS] Jan Janas, Stochel, J , Unbounded Toeplitz operators in the Segal-Bargmann space. II. MR **95m**:47040
- [MvN] F. J. Murray, J. von Neumann, On ring of Operators,IV, *Annals of Mathematics*, **44** (1943), 716-808. MR **5**:101a
- [Pu] L. Pukanszki, The Plancherel formula for the universal covering group of $PSL(2, \mathbb{R})$, *Math Annalen*, **156** (1964), 96-143. MR **30**:1215
- [Pi] G. Pisier, Espaces de Banach quantiques: une introduction à la théorie des espaces des operateurs, *Journée Annuelle, Soc. Math. France*, 1994. MR **98e**:46019
- [Ra1] F. Rădulescu, On the von Neumann Algebra of Toeplitz Operators with Automorphic Symbol, in Subfactors, *Proceedings of the Taniguchi Symposium on Operator Algebras*, eds. H. Araki, Y. Kawahigashi, H. Kosaki, World Scientific, Singapore-New Jersey, 1994. MR **96j**:47024
- [Ra2] F. Rădulescu, Random matrices, amalgamated free products and subfactors in free group factors of noninteger index, *Inv. Math.***115**, pp. 347-389, (1994). MR **95c**:46102
- [Ra3] F. Rădulescu, The arithmetic Hecke operators and Berezin Quantization, *Comptes Rendu Acad. Sci. Paris, Serie I. Mathématique* **322** (1996), no. 6, 541–546. MR **97i**:46112
- [Ra4] F. Rădulescu, The Γ invariant form of the Berezin quantization of the upper halfplane (Preprint 1995, *Memoirs A. M. S.*, 1998. MR **98j**:46080
- [Sa] P. Sally, Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group, *Memoirs A. M. S.*, 1968. MR **38**:3380
- [St] Stout, E. L., On some algebras of analytic functions on finite open Riemann surfaces. *Math. Z.*, **92**, 1966 366–379. MR **34**:358
- [Saf] F. H. Szafraniec, Unbounded subnormal operators, *Bull. Iranian Math. Soc.* **17** (1990), 67–79. MR **91f**:47035
- [Vo] D. Voiculescu, Circular and semicircular systems and free product factors, in *Operator Algebras, Unitary Representations, Enveloping algebras and Invariant Theory*, *Prog. Math.*, **92**, pp. 45-60, Boston: Birkhauser 1990. MR **92e**:46124

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52246
E-mail address: radulesc@math.uiowa.edu