ADJACENCY PRESERVING MAPPINGS
OF INVARIANT SUBSPACES OF A NULL SYSTEM

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Abstract. In the space $I_r$ of invariant $r$-dimensional subspaces of a null system in $(2r+1)$-dimensional projective space, W.L. Chow characterized the basic group of transformations as all the bijections $\varphi : I_r \to I_r$, for which both $\varphi$ and $\varphi^{-1}$ preserve adjacency. In the present paper we show that the two conditions $\varphi : I_r \to I_r$ is a surjection and $\varphi$ preserves adjacency are sufficient to characterize the basic group. At the end of this paper we give an application to Lie geometry.

1. Introduction

Let $n, r$ be positive integers, $3 \leq n = 2r+1$. Let $\Pi$ be an arbitrary $n$-dimensional Pappian projective space. A null system $\delta$ on $\Pi$ is a polarity on $\Pi$ which satisfies $x \in x^\delta$ for every point $x$ of $\Pi$. The space of the $r$-dimensional subspaces of $\Pi$ which are invariant under a fixed null system $\delta$ will be denoted by $I_r := \{a \in [r] \mid a^\delta = a\}$, where $[k], -1 \leq k \leq n$, is the set of all $k$-dimensional subspaces of $\Pi$.

The basic group of transformations in the space $I_r$ (also called the group of semi-symplectic transformations) consists of the transformations induced by all the collineations $f$ of $\Pi$ which satisfy $\delta f = f \delta$. Two invariant $r$-dimensional subspaces $a, b$ are at distance $d$, if their intersection is $(r-d)$-dimensional. If $d = 1$, then they are called adjacent.

W.L. Chow [4] has shown that any bijection $\varphi : I_r \to I_r$, for which both $\varphi$ and $\varphi^{-1}$ preserve adjacency is induced by a collineation of $\Pi$. Observably, any collineation $\varphi$ of $\Pi$ with $\delta \varphi = \varphi \delta$ preserves adjacency in both directions. From a different point of view, L.K. Hua [5], [6] proved the fundamental theorem in the geometry of symmetric matrices under further hypotheses. For a brief history of the development of this problem see Wan [9], [10]. We may consider the theorem of Chow as a Beckman-Quarles type theorem [8] on distance preserving mappings of the space $I_r$. Thus Chow’s theorem may be seen as an early result in the discipline characterizations of geometrical mappings under mild hypotheses [3].

In the present paper we characterize the basic group under mild hypotheses:

**Theorem.** Let $r, n \in \mathbb{N}, 3 \leq n = 2r + 1$. Let $I_r$ be the space of all invariant $r$-dimensional subspaces of a null system $\delta$ in an $n$-dimensional Pappian projective...
space Π. Let \( \varphi : I_r \to I_r \) be a surjection satisfying

\[
\text{(1.1) if } a, b \text{ are adjacent, then } a^\varphi, b^\varphi \text{ are adjacent}
\]

for all \( a, b \in I_r \). Then \( \varphi \) is a transformation of the basic group of \( I_r \).

2. Preliminaries

In this paper, by dimension, intersection, and subspace we understand projective
dimension, intersection, and subspace.

Let \( n, r \) be integers, \( 3 \leq n = 2r + 1 \). Let \( \Pi \) be an arbitrary \( n \)-dimensional Pappian
projective space, and let \( \delta \) be a null system on \( \Pi \). For any subspaces \( a, b \) of \( \Pi \), we
have the following well-known properties: \( \dim a + \dim a^\delta = 2r, (a + b)^\delta = a^\delta \cap b^\delta, (a \cap b)^\delta = a^\delta + b^\delta \), and \( a \subset b \) implies \( b^\delta \subset a^\delta \). For \( -1 \leq k \leq n \) let \([k]\) be the set of
all \( k \)-dimensional subspaces of \( \Pi \). For each \( a \in [k] \), we call \( \pi := a^\delta \in [2r - k] \) the
conjugate of \( a \). An element \( a \in [k] \) is called invariant, if \( a \subset \pi \) or \( \pi \subset a \). Let

\[
I_k := \{ a \in [k] \mid a \text{ is invariant } \}.
\]

Two elements \( a, b \in I_r \) are called adjacent if their intersection has dimension \( r - 1 \).
Let \( a, b \in I_r \). The distance between \( a \) and \( b \) is defined to be \( d(a, b) := r - \dim(a \cap b) \).
If \( a \neq b \), then \( d = d(a, b) \) is the smallest positive integer with the property that
there exists a sequence of \( d + 1 \) invariant and consequently adjacent subspaces
\( a_1 = a, a_2, \ldots, a_{d+1} = b \in I_r \) (see [4]). From this property we obtain the triangle
inequality \( d(a, c) \leq d(a, b) + d(b, c) \) for all \( a, b, c \in I_r \). Let \( P \in I_{r-1}, a \in I_r \). We
define the distance between \( P \) and \( a \) by

\[
d(P, a) := \min \{ d(a, b) \mid b \in P^* \} = r - \dim(P \cap a) - 1
\]

where \( P^* := \{ l \in I_r \mid P \subset l \} \).

A subset \( M \) of \( I_r \) is called a maximal set of adjacent elements in \( I_r \) if any two
distinct elements of \( M \) are adjacent and if there are no other elements of \( I_r \) which
are adjacent to each element of \( M \).

**Lemma 2.1.** A set \( M \subset I_r \) is maximal iff there is a \( P \in I_{r-1} \) with \( M = P^* \).

**Proof.** Assume there exists a “triangle” \( a, b, c \in I_r \), such that \( a, b, c \) are adjacent and \( P := a \cap b \neq b \cap c =: Q \). Then \( H := a + b = b + c = c + a \in [r + 1] \). Since \( P \subset a \), we have \( a = a^\delta \subset P^\delta \in I_{r+1} \) and \( H = a + b = P^\delta \). Similarly, \( H = b + c = Q^\delta \), i.e. \( P^\delta = Q^\delta \) in contradiction to that \( \delta \) is a one-to-one correspondence.

Let \( \varphi : I_r \to I_r \) be a mapping which satisfies \((\square)\). Then \((P^*)^\varphi \) is contained in
a unique maximal set \( Q^* \). We define

\[
(2.1) \quad P^\varphi := Q \quad \text{if} \quad (P^*)^\varphi \subset Q^*.
\]

Furthermore, following the definition of distance, for any \( a, b \in I_r \) we have

\[
(2.2) \quad d(a, b) \geq d(a^\varphi, b^\varphi).
\]

**Lemma 2.2.** For any invariant subspaces \( x \in I_s, y \in I_t, t, s \leq r \), the subspace
\( x + (\pi \cap y) \) is invariant and has dimension \( \leq r \). In the case \( r = t \), \( x + (\pi \cap y) \) has
dimension \( r \).

**Lemma 2.3.** For any \( a \in I_r \) there exists \( b \in I_r \) with \( a \cap b = \emptyset \), i.e. \( d(a, b) = r + 1 \).

For a proof of Lemma 2.2 and Lemma 2.3 see [4].
Lemma 2.4. For any \(a, b \in I_r\), \(a \neq b\), there exists \(c \in I_r\) with \(d(a, c) = r + 1\) and \(d(b, c) < r + 1\).

Proof. Let \(l \in I_r\) with \(a \cap l = \emptyset\). If \(b \cap l \neq \emptyset\), then take \(c := l\). Suppose \(l\) satisfies \(l \cap b = \emptyset\). Let \(k := \text{dim}(a \cap b) < r\). Let \(s\) be an \((r - k - 1)\)-dimensional subspace of \(b\) with \(s + (a \cap b) = b\). Following Lemma 2.2, \(c := s + \langle l \cap l \rangle\) is an invariant element with dimension \(r\). It is clear that \(b \cap c = s\) and \(a \cap b \cap c = \emptyset\). Suppose \(d(a, c) \neq r + 1\). There is an \(x \in a \cap c\). Hence \(x \in c \subset \overline{r}\) and \(s \subset \overline{r}\). On the other hand, \(x \in a \subset \overline{r}\) implies \(a \cap b \subset \overline{r}\), \(b = (a \cap b) + s \subset \overline{r}\) and \(x \in b\), a contradiction to \(a \cap b \cap c = \emptyset\).

Lemma 2.5. Let \(P \in I_{r-1}\), \(a \in I_r\). Then \(d(P, a) = k\) iff there is a uniquely determined \(b \in P^*\) with \(d(a, b) = k\) and \(d(a, l) = k + 1\) for any \(l \in P^* \setminus \{b\}\).

Proof. “\(\Rightarrow\):” Let \(b := P + (a \cap \overline{r}) \in I_r\). Then \(a \cap b = a \cap \overline{r} = a + \overline{r}\); hence \(\text{dim}(a \cap b) = 2r - \text{dim}(a + \overline{r}) = r - k\) and \(d(a, b) = k\). For any \(l \in P^* \setminus \{b\}\) we have \(k \leq d(a, l) \leq d(a, b) + d(b, l) = k + 1\). Let \(l \in P^*\) with \(d(a, l) = k\), i.e. \(\text{dim}(a \cap l) = r - k\). Since \(a \cap l \subset a \cap \overline{r}\) and \(\text{dim}(a \cap l) = \text{dim}(a \cap \overline{r})\), \(a \cap \overline{r} = a \cap l \subset l\). Hence \(l = P + (a \cap \overline{r}) = b\). “\(\Leftarrow\)” follows straightforwardly from the definition.

Lemma 2.6. Let \(a, b \in I_r\) with \(d(a, b) = k\). Then
\[
\min\{d(P, a) \mid P \subset b, P \in I_{r-1}\} = k - 1.
\]
Furthermore, for all \(P \subset b\), \(P \in I_{r-1}\) we have \(d(P, a) = k - 1\) iff \(a \cap b \subset P\).

Proof. From Lemma 2.5 for all \(P \subset b\), \(P \in I_{r-1}\),
\[
k = d(a, b) \in \{d(P, a), d(P, a) + 1\} = \{r - 1 - \text{dim}(P \cap a), r - \text{dim}(P \cap a)\}.
\]
Thus \(a \cap b \subset P\) is equivalent to \(\text{dim}(P \cap a) = \text{dim}(a \cap b) = r - k\) and \(d(P, a) = k - 1\).

Lemma 2.7. For any \(a \in I_r\) and any projective point \(x \in a\), there exists \(b \in I_r\) with \(d(a, b) = r\) and \(a \cap b = \{x\}\).

Proof. Let \(l \in I_r\), \(l \cap a = \emptyset\). Define \(b := x + (\overline{r} \cap l) \in I_r\).

Lemma 2.8. Let \(P \in I_{r-1}\), \(a \in I_r\). Then for any mapping \(\varphi : I_r \to I_r\) satisfying (1.1) we have \(d(P, a) \geq d(P^\varphi, a^\varphi)\) where \(P^\varphi\) is defined in (2.1).

Proof. Let \(b \in P^*\) with \(d(a, b) = d(P, a)\). Then \(b^\varphi \in (P^\varphi)^*\) and
\[
d(P, a) = d(a, b) \geq d(a^\varphi, b^\varphi) \geq d(P^\varphi, a^\varphi).
\]

3. Proof of the theorem

1. If there are \(a, b \in I_r\), \(d(a, b) = r + 1\) and \(d(a^\varphi, b^\varphi) = r\), then for any \(P \in I_{r-1}\), \(P \subset a\) we have \(a^\varphi \cap b^\varphi \subset P^\varphi\).

Proof. For any \(P \in I_{r-1}\), \(P \subset a\) we have \(P^\varphi \subset a^\varphi\). Let \(c \in P^* \setminus \{a\}\) with \(d(b, c) = r\). Then \(d(b^\varphi, c^\varphi) \leq r\) and \(c^\varphi \in (P^\varphi)^*\). Since \(a, c\) are adjacent, also \(a^\varphi, c^\varphi\) are adjacent, and we have \(c^\varphi \neq a^\varphi\). From Lemma 2.5, \(d(P^\varphi, b^\varphi) = r - 1\) and \(a^\varphi \cap b^\varphi \subset P^\varphi\).

2. For any \(a, b \in I_r\), \(d(a^\varphi, b^\varphi) = r\) implies \(d(a, b) = r\).
Proof. Suppose not; then \(d(a, b) = r + 1\). Choose \(c_1, \ldots, c_r \in I_r\) with \(d(c_i^a, a^a) = r\) and such that the set \(\{a^a \cap c_i^a \mid i = 1, \ldots, r\} \cup \{a^a \cap b^a\}\) is a projective basis of \(a^a\). Then \(r \leq d(c_i, a) \leq r + 1\) for all \(i = 1, \ldots, r\). Choose \(Q \subseteq a\), \(Q \subseteq I_{r-1}\) with \(Q \cap c_i = a \cap c_i\) for all \(i = 1, \ldots, r\). In the case \(d(a, c_i) = r + 1\), from 1, \(a^a \cap c_i^a \subset Q^a\). In the other case \(d(a, c_i) = r\) we have \(d(Q^a, c_i^a) \leq d(Q, c_i^a) = r - 1\), so \(a^a \cap c_i^a \subset Q^a\) for all \(i = 1, \ldots, r\). Furthermore, \(a^a \cap b^a \subset Q^a\). This is a contradiction to that \(\{a^a \cap c_i^a \mid i = 1, \ldots, r\} \cup \{a^a \cap b^a\}\) is a basis of \(a^a\).

3. For any \(a \in I_r, \ Q \subseteq I_{r-1}, \ Q \subseteq a^a\), there exists \(P \subseteq I_{r-1}, \ P \subseteq a\) with \(P^a = Q\).

Proof. Choose \(c_1, \ldots, c_r \in I_r\) with \(d(a^a, c_i^a) = r\) such that \(\{a^a \cap c_i^a \mid i = 1, \ldots, r\}\) is a basis of \(Q\). Then for any \(P \subseteq I_{r-1}\) with \(P \subseteq a\), \(a \cap c_i \subseteq P\) implies \(a^a \cap c_i^a \subset P^a\) and \(P^a = Q\).

4. For any \(a, b \in I_r\) with \(d(a, b) = r + 1\), we have \(d(a^a, b^a) = r + 1\).

Proof. We prove 1 by induction. From 2, \(d(a^a, b^a) \neq r\). Let \(d(a^a, b^a) \neq r + 1 - k\) for some \(k \in \{1, \ldots, r\}\). Assume that \(d(a^a, b^a) = r - k\). Let \(Q \subseteq a^a\) with \(d(Q, b^a) = r - k\). Let \(P \subseteq a\) with \(P^a = Q\). Choose \(l \subseteq P^a \setminus \{a\}\) with \(d(l, b) = r + 1\). Then, by Lemma 2.3 \(l^a \subseteq Q^a \setminus \{a^a\}\) implies \(d(l^a, b^a) = r + 1 - k\), a contradiction. Hence \(d(a^a, b^a) \neq r - k\).

5. \(\varphi\) is injective.

Proof. For any \(a \neq b \in I_r\), from Lemma 2.4 there exists \(c \in I_r\) with \(d(a, c) = r + 1\) and \(d(b, c) < r + 1\). Since \(d(a^a, c^a) = r + 1\) and \(d(b^a, c^a) \leq d(b, c) < r + 1\), we have \(a^a \neq b^a\).

6. \(a, b \in I_r\) are adjacent if \(a^a, b^a\) are adjacent.

Proof. Choose \(c \in I_r\) with \(d(a^a, c^a) = r + 1\) and \(d(b^a, c^a) = r\). Denote \(Q := a^a \cap b^a\). Let \(P \subseteq a\), \(P \subseteq I_{r-1}\) with \(P^a = Q\). Let \(l \subseteq P^a\) with \(d(l, c) = r\); then \(l \neq a\). Since \(l^a \subseteq Q^a\), \(r = d(Q, c^a) \leq d(l^a, c^a) \leq d(l, c) = r\). Following Lemma 2.3 we have \(l^a = b^a\). \(\varphi\) is injective, hence \(l = b\). So \(a\) and \(b\) are adjacent.

7. \(\varphi\) is a transformation of the basic group of \(I_r\).

Proof. From 5 and 6, \(\varphi\) is a bijection of \(I_r\), and \(\varphi\) and \(\varphi^{-1}\) preserve adjacency of pairs of elements of \(I_r\). Chow’s theorem completes the proof of the theorem.

4. Application to Lie geometry

Let \(Q\) denote the Lie quadric

\[
x_1x_2 + x_3x_4 + x_5^2 = 0
\]

in the four-dimensional projective space \(\Pi^4(K)\) over a commutative field \(K\), \(\text{ch} K \neq 2\). We call two elements \(X = K(x_1, \ldots, x_5), Y = K(y_1, \ldots, y_5)\) of \(Q\) conjugate if \(X \sim Y : \iff x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + 2x_5y_5 = 0\).

The Lie transformations of \((Q, \sim)\) are defined as bijections \(\alpha\) of \(Q\) satisfying \(X \sim Y\) iff \(X^\alpha \sim Y^\alpha\). Every Lie transformation is induced by a collineation of \(\Pi^4(K)\) (see e.g. 2).

On the three-dimensional projective space \(\Pi^3(K)\) we define a null system \(\delta\) by

\[
P^\delta = \{Q \mid p_1q_3 - p_3q_1 + p_2q_4 - p_4q_2 = 0\}
\]
where \( P = K(p_1, \ldots, p_4), Q = K(q_1, \ldots, q_4) \). There is a one-to-one correspondence \( \gamma \) between the space of invariant lines \( I_1 \) and the Lie quadric \( Q \) which satisfies

\[
\text{a, b are adjacent } \Leftrightarrow \text{a', b' are distinct and conjugate}
\]

for all \( a, b \in I_1 \). This transformation \( \gamma \) is defined as follows. For every line \( a \) of \( \Pi^3(\mathbb{K}) \) consider the Plücker coordinates \( K(a_{12}, a_{13}, a_{14}, a_{42}, a_{23}) \) of \( a \) where \( a_{ij} = p_i q_j - p_j q_i \) for any two distinct points \( P, Q \in a, P = K(p_1, \ldots, p_4), Q = K(q_1, \ldots, q_4) \). Then \( a_{13} = a_{42} \) if, and only if, \( a = a' \), i.e. \( a \in I_1 \). Define \( \gamma : I_1 \to Q \) by \( a \mapsto K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13}) \). \( \gamma \) is a bijection. Furthermore, any distinct \( a, b \in I_1 \) with Plücker coordinates \( K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13}), K(b_{12}, b_{34}, b_{14}, b_{23}, b_{13}) \) are adjacent if, and only if,

\[
\begin{align*}
& a_{12} b_{34} + a_{34} b_{12} + a_{13} b_{42} + a_{42} b_{13} + a_{14} b_{23} + a_{23} b_{14} = 0 \\
\Leftrightarrow & a_{12} b_{34} + a_{34} b_{12} + a_{14} b_{23} + a_{23} b_{14} + 2 a_{13} b_{13} = 0 \\
\Leftrightarrow & a', b' \text{ are conjugate.}
\end{align*}
\]

In the case \((r, n) = (1, 3)\), the theorem implies the following corollary.

**Corollary 4.1.** Let \( \psi : Q \to Q \) be a surjective mapping which takes pairs of distinct conjugate points of \( Q \) to pairs of distinct conjugate points. Then \( \psi \) is a Lie transformation.

**References**


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