ADJACENCY PRESERVING MAPPINGS OF INVARIANT SUBSPACES OF A NULL SYSTEM

WEN-LING HUANG

(Communicated by Christopher Croke)

ABSTRACT. In the space $I_r$ of invariant $r$-dimensional subspaces of a null system in $(2r + 1)$-dimensional projective space, W.L. Chow characterized the basic group of transformations as all the bijections $\varphi : I_r \to I_r$, for which both $\varphi$ and $\varphi^{-1}$ preserve adjacency. In the present paper we show that the two conditions $\varphi : I_r \to I_r$ is a surjection and $\varphi$ preserves adjacency are sufficient to characterize the basic group. At the end of this paper we give an application to Lie geometry.

1. Introduction

Let $n, r$ be positive integers, $3 \leq n = 2r + 1$. Let $\Pi$ be an arbitrary $n$-dimensional Pappian projective space. A null system $\delta$ on $\Pi$ is a polarity on $\Pi$ which satisfies $x^2 = x^\delta$ for every point $x$ of $\Pi$. The space of the $r$-dimensional subspaces of $\Pi$ which are invariant under a fixed null system $\delta$ will be denoted by $I_r := \{ a \in [r] \mid a^\delta = a \}$, where $[k], -1 \leq k \leq n$, is the set of all $k$-dimensional subspaces of $\Pi$.

The basic group of transformations in the space $I_r$ (also called the group of semi-symplectic transformations) consists of the transformations induced by all the collineations $f$ of $\Pi$ which satisfy $f \delta = \delta f$. Two invariant $r$-dimensional subspaces $a, b$ are at distance $d$, if their intersection is $(r - d)$-dimensional. If $d = 1$, then they are called adjacent.

W.L. Chow [3] has shown that any bijection $\varphi : I_r \to I_r$, for which both $\varphi$ and $\varphi^{-1}$ preserve adjacency is induced by a collineation of $\Pi$. Observably, any collineation $\varphi$ of $\Pi$ with $\delta \varphi = \varphi \delta$ preserves adjacency in both directions. From a different point of view, L.K. Hua [4], [5] proved the fundamental theorem in the geometry of symmetric matrices under further hypotheses. For a brief history of the development of this problem see Wan [9], [10]. We may consider the theorem of Chow as a Beckman-Quarles type theorem [8] on distance preserving mappings of the space $I_r$. Thus Chow’s theorem may be seen as an early result in the discipline characterizations of geometrical mappings under mild hypotheses [6].

In the present paper we characterize the basic group under mild hypotheses:

Theorem. Let $r, n \in \mathbb{N}$, $3 \leq n = 2r + 1$. Let $I_r$ be the space of all invariant $r$-dimensional subspaces of a null system $\delta$ in an $n$-dimensional Pappian projective
space $\Pi$. Let $\varphi : I_r \to I_r$ be a surjection satisfying
\begin{equation}
(1.1) \quad \text{if } a, b \text{ are adjacent, then } a^\varphi, b^\varphi \text{ are adjacent}
\end{equation}
for all $a, b \in I_r$. Then $\varphi$ is a transformation of the basic group of $I_r$.

2. Preliminaries

In this paper, by dimension, intersection, and subspace we understand projective dimension, intersection, and subspace.

Let $n, r$ be integers, $3 \leq n = 2r + 1$. Let $\Pi$ be an arbitrary $n$-dimensional Pappian projective space, and let $\delta$ be a null system on $\Pi$. For any subspaces $a, b$ of $\Pi$, we have the following well-known properties: $\dim a + \dim a^\delta = 2r$, $(a \cap b)^\delta = a^\delta \cap b^\delta$, $(a \cap b)^\delta = a^\delta + b^\delta$, and $a \cap b$ implies $b^\delta \subset a^\delta$. For $-1 \leq k < n$ let $[k]$ be the set of all $k$-dimensional subspaces of $\Pi$. For each $a \in [k]$, we call $\pi := a^\delta \in [2r - k]$ the conjugate of $a$. An element $a \in [k]$ is called invariant, if $a \subset \pi$ or $\pi \subset a$. Let

$$I_k := \{ a \in [k] \mid a \text{ is invariant } \}.$$ 

Two elements $a, b \in I_r$ are called adjacent if their intersection has dimension $r - 1$. Let $a, b \in I_r$. The distance between $a$ and $b$ is defined to be $d(a, b) := r - \dim(a \cap b)$. If $a \neq b$, then $d = d(a, b)$ is the smallest positive integer with the property that there exists a sequence of $d + 1$ invariant and consecutively adjacent subspaces $a_1 = a, a_2, \ldots, a_{d+1} = b \in I_r$ (see [4]). From this property we obtain the triangle inequality $d(a, c) \leq d(a, b) + d(b, c)$ for all $a, b, c \in I_r$. Let $P \in I_{r-1}, a \in I_r$. We define the distance between $P$ and $a$ by

$$d(P, a) := \min \{ d(a, b) \mid b \in P^* \} = r - \dim(P \cap a) - 1$$

where $P^* := \{ l \in I_r \mid P \subset l \}$.

A subset $M$ of $I_r$ is called a maximal set of adjacent elements in $I_r$ if any two distinct elements of $M$ are adjacent and if there are no other elements of $I_r$ which are adjacent to each element of $M$.

Lemma 2.1. A set $M \subset I_r$ is maximal iff there is a $P \in I_{r-1}$ with $M = P^*$.

Proof. Assume there exists a “triangle” $a, b, c \in I_r$, such that $a, b, c, c, a$, are adjacent and $P := a \cap b \neq b \cap c =: Q$. Then $H := a + b = b + c = c + a \in [r + 1]$. Since $P \subset a$, we have $a = a^\delta \subset P^\delta \in I_{r+1}$ and $H = a + b = P^\delta$. Similarly, $H = b + c = Q^\delta$, i.e. $P^\delta = Q^\delta$ in contradiction to that $\delta$ is a one-to-one correspondence.

Let $\varphi : I_r \to I_r$ be a mapping which satisfies $\square$. Then $(P^*)^\varphi$ is contained in a unique maximal set $Q^*$. We define
\begin{equation}
(2.1) \quad P^\varphi := Q \quad \text{if} \quad (P^*)^\varphi \subset Q^*.
\end{equation}

Furthermore, following the definition of distance, for any $a, b \in I_r$ we have
\begin{equation}
(2.2) \quad d(a, b) \geq d(a^\varphi, b^\varphi).
\end{equation}

Lemma 2.2. For any invariant subspaces $x \in I_s, y \in I_t$, $t, s \leq r$, the subspace $x + (\pi \cap y)$ is invariant and has dimension $\leq r$. In the case $r = t$, $x + (\pi \cap y)$ has dimension $r$.

Lemma 2.3. For any $a \in I_r$ there exists $b \in I_r$ with $a \cap b = \emptyset$, i.e. $d(a, b) = r + 1$.

For a proof of Lemma 2.2 and Lemma 2.3 see [4].
Lemma 2.4. For any $a, b \in I_r$, $a \neq b$, there exists $c \in I_r$ with $d(a, c) = r + 1$ and $d(b, c) < r + 1$.

Proof. Let $l \in I_r$ with $a \cap l = \emptyset$. If $b \cap l \neq \emptyset$, then take $c := l$. Suppose $l$ satisfies $l \cap b = \emptyset$. Let $k := \dim(a \cap b) < r$. Let $s$ be an $(r - k - 1)$-dimensional subspace of $b$ with $s + (a \cap b) = b$. Following Lemma 2.2, $c := s + (\mathcal{F} \cap l)$ is an invariant element with dimension $r$. It is clear that $b \cap c = s$ and $a \cap b \cap c = \emptyset$. Suppose $d(a, c) \neq r + 1$. There is an $x \in a \cap c$. Hence $x \in c \subset \mathcal{F}$, and $c \subset \mathcal{F}$. On the other hand, $x \in a \cap c$ implies $a \cap b \subset \mathcal{F}$, $b = (a \cap b) + s \subset \mathcal{F}$ and $x \in b$, a contradiction to $a \cap b \cap c = \emptyset$. □

Lemma 2.5. Let $P \in I_{r-1}$, $a \in I_r$. Then $d(P, a) = k$ iff there is a uniquely determined $b \in P^*$ with $d(a, b) = k$ and $d(a, l) = k + 1$ for any $l \in P^* \setminus \{b\}$.

Proof. “⇒” Let $b := P + (a \cap \mathcal{F}) \in I_r$. Then $a \cap b = a \cap \mathcal{F} = a + \mathcal{F}$, hence $\dim(a \cap b) = 2r - \dim(a + \mathcal{F}) = r - k$ and $d(a, b) = k$. For any $l \in P^* \setminus \{b\}$ we have $k \leq d(a, l) \leq d(a, b) + d(l, b) = k + 1$. Let $l \in P^*$ with $d(a, l) = k$, i.e. $\dim(a \cap b) = r - k$. Since $a \cap l \subset a \cap \mathcal{F}$ and $\dim(a \cap l) = \dim(a \cap \mathcal{F})$, $a \cap \mathcal{F} = a \cap l \subset l$. Hence $l = P + (a \cap \mathcal{F}) = b$. “⇐” follows straightforwardly from the definition. □

Lemma 2.6. Let $a, b \in I_r$ with $d(a, b) = k$. Then

$$\min\{d(P, a) \mid P \subset b, P \in I_{r-1}\} = k - 1.$$

Furthermore, for all $P \subset b$, $P \in I_{r-1}$ we have $d(P, a) = k - 1$ iff $a \cap b \subset P$.

Proof. From Lemma 2.5 for all $P \subset b$, $P \in I_{r-1}$,

$$k = d(a, b) \in \{d(P, a), d(P, a) + 1\} = \{r - 1 - \dim(P \cap a), r - \dim(P \cap a)\}.$$

Thus $a \cap b \subset P$ is equivalent to $\dim(P \cap a) = \dim(a \cap b) = r - k$ and to $d(P, a) = k - 1$. □

Lemma 2.7. For any $a \in I_r$ and any projective point $x \in a$, there exists $b \in I_r$ with $d(a, b) = r$ and $a \cap b = \{x\}$.

Proof. Let $l \in I_r$, $l \cap a = \emptyset$. Define $b := x + (\mathcal{F} \cap l) \in I_r$. □

Lemma 2.8. Let $P \in I_{r-1}, a \in I_r$. Then for any mapping $\varphi : I_r \rightarrow I_r$ satisfying (1.1) we have $d(P, a) \geq d(P^\varphi, a^\varphi)$ where $P^\varphi$ is defined in (2.1).

Proof. Let $b \in P^*$ with $d(a, b) = d(P, a)$. Then $b^\varphi \in (P^\varphi)^*$ and

$$d(P, a) = d(a, b) \geq d(a^\varphi, b^\varphi) \geq d(P^\varphi, a^\varphi).$$

□

3. Proof of the theorem

1. If there are $a, b \in I_r$ with $d(a, b) = r + 1$ and $d(a^\varphi, b^\varphi) = r$, then for any $P \in I_{r-1}$, $P \subset a$ we have $a^\varphi \cap b^\varphi \subset P^\varphi$.

Proof. For any $P \in I_{r-1}$, $P \subset a$ we have $P^\varphi \subset a^\varphi$. Let $c \in P^* \setminus \{a\}$ with $d(b, c) = r$. Then $d(b^\varphi, c^\varphi) \leq r$ and $c^\varphi \in (P^\varphi)^*$. Since $a, c$ are adjacent, also $a^\varphi, c^\varphi$ are adjacent, and we have $c^\varphi \neq a^\varphi$. From Lemma 2.5, $d(P^\varphi, b^\varphi) = r - 1$ and $a^\varphi \cap b^\varphi \subset P^\varphi$. □

2. For any $a, b \in I_r$, $d(a^\varphi, b^\varphi) = r$ implies $d(a, b) = r$.
Proof. Suppose not; then \(d(a, b) = r + 1\). Choose \(c_1, \ldots, c_r \in I_r\) with \(d(c_i^a, a^c) = r\) and such that the set \(\{a^c \cap c_i^a \mid i = 1, \ldots, r\}\) is a projective basis of \(a^c\). Then \(r \leq d(c_i, a) \leq r + 1\) for all \(i = 1, \ldots, r\). Choose \(Q \subset a, Q \in I_{r-1}\) with \(Q \cap c_i = a \cap c_i\) for all \(i = 1, \ldots, r\). In the case \(d(a, c_i) = r + 1\), from (1), \(a^c \cap c_i^a \subset Q^c\). In the other case \(d(a, c_i) = r\) we have \(d(Q^c, c_i^a) \leq d(Q, c_i) = r - 1\), so \(a^c \cap c_i^a \subset Q^c\) for all \(i = 1, \ldots, r\). Furthermore, \(a^c \cap b^c \subset Q^c\). This is a contradiction to that \(\{a^c \cap c_i^a \mid i = 1, \ldots, r\}\) is a basis of \(a^c\).

3. For any \(a \in I_r, Q \in I_{r-1}, Q \subset a^c\), there exists \(P \in I_{r-1}, P \subset a\) with \(P^c = Q\).

Proof. Choose \(c_1, \ldots, c_r \in I_r\) with \(d(a^c, c_i^a) = r\) such that \(\{a^c \cap c_i^a \mid i = 1, \ldots, r\}\) is a basis of \(Q\). Then for any \(P \in I_{r-1}\) with \(P \subset a, a \cap c_i \subset P\) implies \(a^c \cap c_i^a \subset P^a\) and \(P^c = Q\).

4. For any \(a, b \in I_r\) with \(d(a, b) = r + 1\), we have \(d(a^c, b^c) = r + 1\).

Proof. We prove (1) by induction. From (2), \(d(a^c, b^c) \neq r\). Let \(d(a^c, b^c) \neq r + 1 - k\) for some \(k \in \{1, \ldots, r\}\). Assume that \(d(a^c, b^c) = r - k\). Let \(Q \subset a^c\) with \(d(Q, b^c) = r - k\). Let \(P \subset a\) with \(P^c = Q\). Choose \(l \in P^a \setminus \{a\}\) with \(d(l, b) = r + 1\). Then, by Lemma (2.3) \(l^c \in P^a \setminus \{a\}\) implies \(d(l^c, b^c) = r + 1 - k\), a contradiction. Hence \(d(a^c, b^c) = r - k\).

5. \(\varphi\) is injective.

Proof. For any \(a \neq b \in I_r\), from Lemma (2.4) there exists \(c \in I_r\) with \(d(a, c) = r + 1\) and \(d(b, c) < r + 1\). Since \(d(a^c, c^a) = r + 1\) and \(d(b^c, c^a) \leq d(b, c) < r + 1\), we have \(a^c \neq b^c\).

6. \(a, b \in I_r\) are adjacent if \(a^c, b^c\) are adjacent.

Proof. Choose \(c \in I_r\) with \(d(a^c, c^a) = r + 1\) and \(d(b^c, c^a) = r\). Denote \(Q := a^c \cap b^c\). Let \(P \subset a, P \in I_{r-1}\) with \(P^c = Q\). Let \(l \in P^a\) with \(d(l, c) = r\); then \(l \neq a\). Since \(l^c \in Q^a\), \(r = d(Q, c^a) \leq d(l^c, c^a) \leq d(l, c) = r\). Following Lemma (2.3) we have \(l^c = b^c\). \(\varphi\) is injective, hence \(l = b\). So \(a\) and \(b\) are adjacent.

7. \(\varphi\) is a transformation of the basic group of \(I_r\).

Proof. From (5) and (6), \(\varphi\) is a bijection of \(I_r\), and \(\varphi\) and \(\varphi^{-1}\) preserve adjacency of pairs of elements of \(I_r\). Chow’s theorem completes the proof of the theorem.

4. Application to Lie geometry

Let \(Q\) denote the Lie quadric
\[
x_1x_2 + x_3x_4 + x_5^2 = 0
\]
in the four-dimensional projective space \(\Pi^4(K)\) over a commutative field \(K, \text{ch } K \neq 2\). We call two elements \(X = K(x_1, \ldots, x_5), Y = K(y_1, \ldots, y_5)\) of \(Q\) conjugate if \(X \sim Y \iff x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + 2x_5y_5 = 0\).

The Lie transformations of \((Q, \sim)\) are defined as bijections \(\alpha\) of \(Q\) satisfying \(X \sim Y\) if \(X^\alpha \sim Y^\alpha\). Every Lie transformation is induced by a collineation of \(\Pi^4(K)\) (see e.g. (2)).

On the three-dimensional projective space \(\Pi^3(K)\) we define a null system \(\delta\) by
\[
P^\delta = \{Q \mid p_1q_3 - p_3q_1 + p_2q_4 - p_4q_2 = 0\}\]
where $P = K(p_1, \ldots, p_4)$, $Q = K(q_1, \ldots, q_4)$. There is a one-to-one correspondence
$
\gamma
$
between the space of invariant lines $I_1$ and the Lie quadric $Q$ which satisfies

$$a, b \text{ are adjacent } \iff a^\gamma, b^\gamma \text{ are distinct and conjugate}
$$

for all $a, b \in I_1$. This transformation $\gamma$ is defined as follows. For every line $a$ of $\Pi(K)$ consider the Plücker coordinates $K(a_{12}, a_{13}, a_{14}, a_{34}, a_{23})$ of $a$ where $a_{ij} = p_iq_j - p_jq_i$ for any two distinct points $P, Q \in a$, $P = K(p_1, \ldots, p_4)$, $Q = K(q_1, \ldots, q_4)$. Then $a_{13} = a_{42}$ if, and only if, $a = a^\gamma$, i.e. $a \in I_1$. Define $\gamma : I_1 \rightarrow Q$ by $a \mapsto K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13})$. $\gamma$ is a bijection. Furthermore, any distinct $a, b \in I_1$ with Plücker coordinates $K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13})$, $K(b_{12}, b_{34}, b_{14}, b_{23}, b_{13})$ are adjacent if, and only if,

$$a_{12}b_{34} + a_{34}b_{12} + a_{13}b_{42} + a_{42}b_{13} + a_{14}b_{23} + a_{23}b_{14} = 0$$

$$a_{12}b_{34} + a_{34}b_{12} + a_{14}b_{23} + a_{23}b_{14} + 2a_{13}b_{13} = 0$$

$$a^\gamma, b^\gamma \text{ are conjugate.}
$$

In the case $(r, n) = (1, 3)$, the theorem implies the following corollary.

**Corollary 4.1.** Let $\psi : Q \rightarrow Q$ be a surjective mapping which takes pairs of distinct conjugate points of $Q$ to pairs of distinct conjugate points. Then $\psi$ is a Lie transformation.

**References**


*Mathematisches Seminar, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany*

*E-mail address: huang@math.uni-hamburg.de*