LIOUVILLE NUMBERS, RAJCHMAN MEASURES, AND SMALL CANTOR SETS

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Abstract. We show that the set of Liouville numbers carries a positive measure whose Fourier transform vanishes at infinity. The proof is based on a new construction of a Cantor set of Hausdorff dimension zero supporting such a measure.

1. Introduction

In the year 1844 JOSEPH LIOUVILLE constructed an interesting class of transcendental numbers, namely

\[ \mathbb{L} = \{ x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N} : \exists q \in \mathbb{N} : \|qx\| < q^{-n} \}, \]

now called the set of Liouville numbers. Here \( \|x\| = \min_{m \in \mathbb{Z}} |x - m| \) denotes the distance of a real number \( x \) to the nearest integer. For example, the number \( \sum 10^{-k!} = 0.110001000... \) (where the 1 is only in places \( n! \)) belongs to \( \mathbb{L} \). From the well known theorem of JARNIK [4] and BESICOVITCH [1] it follows immediately that \( \mathbb{L} \) has Hausdorff dimension zero, so we consider \( \mathbb{L} \) to be a ‘small’ set.

In this note we show that \( \mathbb{L} \) supports a positive measure whose Fourier transform vanishes at infinity. Such measures are called Rajchman measures; see the survey article by LYONS [5] for references. A detailed discussion of related constructions can be found in KÖRNER’s paper [3].

The proof of our result is based on a new construction of a Cantor set with Hausdorff dimension zero carrying a Rajchman measure.

2. Main results

In the sequel \( \mathbb{P}_M \) denotes the set of prime numbers between \( M \) and \( 2M \) where \( M \) is a positive integer. We choose a sequence of positive integers \( (M_k)_{k \in \mathbb{N}} \) with \( M_1 < 2M_1 < M_2 < 2M_2 < M_3 < 2M_3 < \cdots \) and define the set

\[ S_\infty = \bigcap_{k=1}^{\infty} \bigcup_{p \in \mathbb{P}_{M_k}} \{ x \in [0, 1] : \|px\| \leq p^{-1-k} \}, \]

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In fact, $S_\infty$ is compact because $\overline{E_k(p)} = \{ x \in [0,1] : \|px\| \leq p^{-1-k} \}$ equals

$$[0, p^{-2-k}] \cup \bigcup_{m=1}^{p-1} \left[ \frac{m}{p} - p^{-2-k}, \frac{m}{p} + p^{-2-k} \right] \cup [1 - p^{-2-k}, 1].$$

Proposition 2.1. $S_\infty$ is a Cantor set of Hausdorff dimension zero.

Proof. According to (2.1) the set $E_k(p)$ can be covered by $p + 1$ intervals of length $\leq 2p^{-2-k}$. For every $k \in \mathbb{N}$ and $\alpha = 3/(2 + k)$ this implies

$$H^\alpha(S_\infty) \leq \sum_{p \in \mathbb{P}_M} (p + 1) (2p^{-2-k})^{3/(2+k)} < \infty.$$

Therefore, $S_\infty$ has Hausdorff dimension zero.

For the following theorem recall that the Fourier transform of a positive bounded measure $\mu$ is defined by

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{-2\pi i xt} d\mu(t) \quad (x \in \mathbb{R}).$$

Theorem 2.2. There exists a sequence $(M_k)_{k \in \mathbb{N}}$ such that the corresponding set $S_\infty$ supports a positive measure $\mu_\infty$ with

$$\lim_{|x| \to \infty} \hat{\mu}_\infty(x) = 0.$$

As an application we obtain

Theorem 2.3. The set $S_\infty \setminus \mathbb{Q}$ is contained in $L$. Therefore, the set of Liouville numbers carries a Rajchman measure.

Proof. From the definition of $L$ and $S_\infty$ it is obvious that $S_\infty \setminus \mathbb{Q} \subset L$. The Rajchman measure $\mu_\infty$ is supported by $S_\infty$. Removing the rational points from $S_\infty$ means removing a zero set from the support of $\mu_\infty$. By a simple regularity argument we can replace $\text{supp}(\mu_\infty) \setminus \mathbb{Q}$ by a compact set with positive $\mu_\infty$-measure.

3. Proof of Theorem 2.2

Proof. The proof of Theorem 2.2 is based on a modification of a construction elaborated in [2]. There (in section 3) we constructed, for given positive $\alpha > 0$ and positive integers $M$, certain 1-periodic functions $g_M \in C^2(\mathbb{R})$ with

(3.1) $\text{supp}(g_M) \subseteq \bigcup_{p \in \mathbb{P}_M} \{ x : \|px\| \leq p^{-1-\alpha} \}$ and $\hat{g}_M(0) = 1$.

Moreover, these functions $g_M$ had the following nice property ([2], Lemma 3.2):

Lemma 3.1. For every $\psi \in C^2(\mathbb{R})$ and $\delta > 0$ there exists $M_0 = M_0(\psi, \delta)$ s.t.

$$\left| \hat{\psi}(x) - \hat{\psi}_M(x) \right| \leq \delta \cdot \theta(x) \quad \forall x \in \mathbb{R}$$

for all $M \geq M_0$, where $\theta(x) = (1 + |x|)^{1/(2+\alpha)} \cdot \log (e + |x|) \cdot \log (e + \log (e + |x|))$.

We will apply Lemma 3.1 to our situation by replacing $\alpha$ by $k$ and $\delta$ by $\theta_k(x) = (1 + |x|)^{-1/(2+k)} \cdot \log (e + |x|) \cdot \log (e + \log (e + |x|)) \quad (k \in \mathbb{N})$. 

To be more precise, we fix an initial function \( \psi_0 : \mathbb{R}_+ \to \mathbb{R}_+ \) with
\[
\psi_0 \in C_c^2(\mathbb{R}), \quad \int \psi_0(x)dx = 1, \quad \psi_0|_{[0,1]} > 0, \quad \text{and} \quad \psi_0|_{\mathbb{R}\setminus[0,1]} = 0.
\]
Next we define a sequence \( (\tau_k)_{k \in \mathbb{N}} \) by \( \tau_k = (\max_{x \in \mathbb{R}} \theta(x))^{-1} \). As a first step we replace \( \alpha \) in the setting above by \( k = 1 \). According to Lemma 3.1, we can find a positive integer \( M_1 = M_1(\psi_0, \tau_1 3^{-1}) \) such that
\[
|\hat{\psi_{M_1}}^\wedge (x) - \hat{\psi}(x)| \leq \tau_1 3^{-1} \theta_1(x) \quad \forall \ x \in \mathbb{R}.
\]
Now we repeat the same procedure, but this time replacing \( \alpha \) by \( k = 2, \theta_1 \) by \( \theta_2 \), and \( \psi \) by \( \psi_{g_{M_1}} \). Then again Lemma 3.1 implies the existence of an integer \( M_2 = M_2(\psi_{g_{M_1}}, \tau_2 3^{-2}) \) such that
\[
|\hat{\psi_{g_{M_1},g_{M_2}}}^\wedge (x) - \hat{\psi_{g_{M_1}}}^\wedge (x)| \leq \tau_2 3^{-2} \theta_2(x) \quad \forall \ x \in \mathbb{R}.
\]
By repeating this process we obtain for every index \( k \) an integer
\[
M_k = M_k(\psi_{g_{M_1},g_{M_2},\ldots,g_{M_{k-1}}}, \tau_k 3^{-k})(k \in \mathbb{N}),
\]
fulfilling the corresponding estimation.

Now we assume \( S_\infty \) to be constructed according to \( (M_k)_{k \in \mathbb{N}} \). We set
\[
G_0 = 1, \quad \text{and} \quad G_k = g_{M_1} \cdots g_{M_k}(k \in \mathbb{N}).
\]
By Lemma 3.1, we obtain for every \( k \in \mathbb{N}_0 \) and all \( x \in \mathbb{R} \)
\[
|\hat{\psi_{G_{k+1}}}^\wedge (x) - \hat{\psi_{G_k}}^\wedge (x)| \leq \tau_{k+1} 3^{-k-1} \theta_{k+1}(x).
\]
Let \( \lambda \) be Lebesgue measure and define a sequence of measures by
\[
\mu_k = \psi_{G_k} \lambda \quad (k \in \mathbb{N}_0).
\]
Because of (3.2), the sequence \( (\mu_k)_{k \in \mathbb{N}_0} \) is a Cauchy sequence w.r.t. the supremum norm. Taking \( g_{M_k}(0) = 1 \) into account (see (3.1)), we conclude the weak convergence of \( (\mu_k) \) to a bounded measure \( \mu_\infty \) (Lévy’s continuity theorem). Moreover, by (3.2) and a geometric series estimate we get \( \|\mu_\infty(0) - \hat{\psi}(0)\| \leq \frac{1}{2} \), so that \( \mu_\infty \) has at least mass \( \frac{1}{2} \). The claimed Fourier asymptotic of \( \mu_\infty \) follows easily from (3.2) and a simple geometric series argument, also taking into account that \( \hat{\mu}_p(x) = O \left( |x|^{-2} \right) \) for fixed \( p \). The construction of \( \mu_\infty \) is based on successive multiplication of densities \( g_{M_k} \). Therefore, by (3.1) it is clear that the support of \( \mu_\infty \) must be contained in the Cantor set \( S_\infty \). This concludes the proof of Theorem 2.2.

Remark 3.2. Why prime numbers? Let us sketch the answer. The proof of Lemma 5.1 (2, section 4) rests on the prime number theorem \#\mathbb{P}_M \sim M/\log M \) (Hardy and Wright 3 (22.19.3)). So it is clear that although the number of primes between \( M \) and \( 2M \) is strictly increasing with \( M \), the primes are somehow ‘thinning out’ at infinity. This observation is of great importance in the proof of Lemma 3.1 when one tries to allow the ‘\( \delta \)’ in the estimation to become arbitrarily small.

References


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