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LIOUVILLE NUMBERS, RAJCHMAN MEASURES, AND SMALL CANTOR SETS

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Abstract. We show that the set of Liouville numbers carries a positive measure whose Fourier transform vanishes at infinity. The proof is based on a new construction of a Cantor set of Hausdorff dimension zero supporting such a measure.

1. Introduction

In the year 1844 Joseph Liouville constructed an interesting class of transcendental numbers, namely

$$\mathbb{L} = \left\{ x \in \mathbb{R}\backslash\mathbb{Q} \;:\; \forall\, n \in \mathbb{N}:\; \exists\, q \in \mathbb{N}:\; \|qx\| < q^{-n} \right\},$$

now called the set of *Liouville numbers*. Here $\|x\| = \min_{m\in\mathbb{Z}} |x - m|$ denotes the distance of a real number x to the nearest integer. For example, the number $\sum 10^{-k!} = 0.110001000...$ (where the 1 is only in places $n!$) belongs to $\mathbb{L}$. From the well known theorem of Jarnik [4] and Besicovitch [1] it follows immediately that $\mathbb{L}$ has Hausdorff dimension zero, so we consider $\mathbb{L}$ to be a 'small' set.

In this note we show that $\mathbb{L}$ supports a positive measure whose Fourier transform vanishes at infinity. Such measures are called *Rajchman measures*; see the survey article by Lyons [6] for references. A detailed discussion of related constructions can be found in Körner's paper [5].

The proof of our result is based on a new construction of a Cantor set with Hausdorff dimension zero carrying a Rajchman measure.

2. Main results

In the sequel $\mathbb{P}_M$ denotes the set of prime numbers between M and $2M$ where M is a positive integer. We choose a sequence of positive integers $(M_k)_{k\in\mathbb{N}}$ with $M_1 < 2M_1 < M_2 < 2M_2 < M_3 < 2M_3 < \cdots$ and define the set

$$S_\infty = \bigcap_{k=1}^{\infty} \bigcup_{p\in\mathbb{P}_{M_k}} \left\{ x \in [0,1] : \|px\| \le p^{-1-k} \right\}.$$

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In fact, S_∞ is compact because $\overline{E}_k(p) = \left\{x \in [0,1] : \|px\| \le p^{-1-k}\right\}$ equals

$$[0, p^{-2-k}] \cup \bigcup_{m=1}^{p-1} \left[\frac{m}{p} - p^{-2-k}, \frac{m}{p} + p^{-2-k}\right] \cup [1 - p^{-2-k}, 1]\,. \tag{2.1}$$

Proposition 2.1. *S_∞ is a Cantor set of Hausdorff dimension zero.*

Proof. According to (2.1) the set $\overline{E}_k(p)$ can be covered by $p+1$ intervals of length $\le 2p^{-2-k}$. For every $k \in \mathbb{N}$ and $\alpha = 3/(2+k)$ this implies

$$H^\alpha(S_\infty) \le \sum_{p \in \mathbb{P}_{M_k}} (p+1)\left(2p^{-2-k}\right)^{3/(2+k)} < \infty.$$

Therefore, S_∞ has Hausdorff dimension zero. □

For the following theorem recall that the Fourier transform of a positive bounded measure μ is defined by

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{-2\pi i x t} d\mu(t) \qquad (x \in \mathbb{R}).$$

Theorem 2.2. *There exists a sequence $(M_k)_{k \in \mathbb{N}}$ such that the corresponding set S_∞ supports a positive measure μ_∞ with*

$$\lim_{|x| \to \infty} \hat{\mu}_\infty(x) = 0.$$

As an application we obtain

Theorem 2.3. *The set $S_\infty \backslash \mathbb{Q}$ is contained in $\mathbb{L}$. Therefore, the set of Liouville numbers carries a Rajchman measure.*

Proof. From the definition of $\mathbb{L}$ and S_∞ it is obvious that $S_\infty \backslash \mathbb{Q} \subset \mathbb{L}$. The Rajchman measure μ_∞ is supported by S_∞. Removing the rational points from S_∞ means removing a zero set from the support of μ_∞. By a simple regularity argument we can replace $\mathrm{supp}(\mu_\infty) \backslash \mathbb{Q}$ by a compact set with positive μ_∞-measure. □

3. Proof of Theorem 2.2

Proof. The proof of Theorem 2.2 is based on a modification of a construction elaborated in [2]. There (in section 3) we constructed, for given positive $\alpha > 0$ and positive integers M, certain 1-periodic functions $g_M \in C^2(\mathbb{R})$ with

$$\mathrm{supp}(g_M) \subseteq \bigcup_{p \in \mathbb{P}_M} \left\{x : \|px\| \le p^{-1-\alpha}\right\} \quad \text{and} \quad \hat{g}_M(0) = 1. \tag{3.1}$$

Moreover, these functions g_M had the following nice property ([2], Lemma 3.2):

Lemma 3.1. *For every $\psi \in C_c^2(\mathbb{R})$ and $\delta > 0$ there exists $M_0 = M_0(\psi, \delta)$ s.t.*

$$\left|[\psi g_M]^\wedge(x) - \hat{\psi}(x)\right| \le \delta \cdot \theta(x) \qquad \forall\, x \in \mathbb{R}$$

for all $M \ge M_0$, where $\theta(x) = (1+|x|)^{-1/(2+\alpha)} \cdot \log(e+|x|) \cdot \log(e + \log(e+|x|))$.

We will apply Lemma 3.1 to our situation by replacing α by k and θ by

$$\theta_k(x) = (1+|x|)^{-1/(2+k)} \cdot \log(e+|x|) \cdot \log(e+\log(e+|x|)) \quad (k \in \mathbb{N}).$$

To be more precise, we fix an initial function $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\psi_0 \in C_c^2(\mathbb{R}), \quad \int \psi_0(x)dx = 1, \quad \psi_0|_{]0,1[} > 0, \quad \text{and} \quad \psi_0|_{\mathbb{R}\setminus[0,1]} \equiv 0.$$

Next we define a sequence $(\tau_k)_{k\in\mathbb{N}}$ by $\tau_k = (\max_{x\in\mathbb{R}} \theta_k(x))^{-1}$. As a first step we replace α in the setting above by $k = 1$. According to Lemma 3.1 we can find a positive integer $M_1 = M_1 \left(\psi_0, \tau_1 3^{-1}\right)$ such that

$$\left| [\psi g_{M_1}]^\wedge (x) - \hat{\psi}(x) \right| \leq \tau_1 3^{-1} \theta_1(x) \qquad \forall\ x \in \mathbb{R}.$$

Now we repeat the same procedure, but this time replacing α by $k = 2$, θ_1 by θ_2, and ψ by $\psi_0 g_{M_1}$. Then again Lemma 3.1 implies the existence of an integer $M_2 = M_2 \left(\psi_0 g_{M_1}, \tau_2 3^{-2}\right)$ such that

$$\left| [\psi g_{M_1} g_{M_2}]^\wedge (x) - [\psi_0 g_{M_1}]^\wedge(x) \right| \leq \tau_2 3^{-2} \theta_2(x) \qquad \forall\ x \in \mathbb{R}.$$

By repeating this process we obtain for every index k an integer

$$M_k \ = \ M_k \left(\psi_0 g_{M_1} g_{M_2} \cdots g_{M_{k-1}}, \tau_k 3^{-k}\right) \qquad (k \in \mathbb{N}),$$

fulfilling the corresponding estimation.

Now we assume S_∞ to be constructed according to $(M_k)_{k\in\mathbb{N}}$. We set

$$G_0 = 1, \quad \text{and} \quad G_k = g_{M_1} \cdots g_{M_k} \qquad (k \in \mathbb{N}).$$

By Lemma 3.1 we obtain for every $k \in \mathbb{N}_0$ and all $x \in \mathbb{R}$

$$|[\psi_0 G_{k+1}]^\wedge(x) - [\psi_0 G_k]^\wedge(x)| \ \leq \ \tau_{k+1} 3^{-k-1} \theta_{k+1}(x). \tag{3.2}$$

Let λ be Lebesgue measure and define a sequence of measures by

$$\mu_k = \psi_0 G_k \lambda \qquad (k \in \mathbb{N}_0).$$

Because of (3.2) the sequence $(\hat{\mu}_k)_{k\in\mathbb{N}_0}$ is a Cauchy sequence w.r.t. the supremum norm. Taking $\hat{g}_{M_k}(0) = 1$ into account (see (3.1)), we conclude the weak convergence of $(\mu_k)_k$ to a bounded measure μ_∞ (Lévy's continuity theorem). Moreover, by (3.2) and a geometric series estimate we get $|\hat{\mu}_\infty(0) - \hat{\psi}(0)| \leq \frac{1}{2}$, so that μ_∞ has at least mass $\frac{1}{2}$. The claimed Fourier asymptotic of μ_∞ follows easily from (3.2) and a simple geometric series argument, also taking into account that $\hat{\mu}_p(x) = O\left(|x|^{-2}\right)$ for fixed p. The construction of μ_∞ is based on successive multiplication of densities g_{M_k}. Therefore, by (3.1) it is clear that the support of μ_∞ must be contained in the Cantor set S_∞. This concludes the proof of Theorem 2.2. □

Remark 3.2. Why prime numbers? Let us sketch the answer. The proof of Lemma 3.1 ([2], section 4) rests on the prime number theorem $\#\mathbb{P}_M \sim M/\log M$ (Hardy and Wright [3] (22.19.3)). So it is clear that although the number of primes between M and $2M$ is strictly increasing with M, the primes are somehow 'thinning out' at infinity. This observation is of great importance in the proof of Lemma 3.1, when one tries to allow the 'δ' in the estimation to become arbitrarily small.

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