CARDINAL SPLINE INTERPOLATION FROM $H^1(\mathbb{Z})$ TO $L_1(\mathbb{R})$

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ABSTRACT. Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those sequences $y = \{y_j\}_{j \in \mathbb{Z}} \in l_1(\mathbb{Z})$, such that $Hy = \{Hy_j\} \in l_1(\mathbb{Z})$, where $Hy_j = \sum_{k \neq j} (k - j)^{-1} y_j$, $j \in \mathbb{Z}$, is the discrete Hilbert transform of $y$. For a sequence $y = \{y_j\}_{j \in \mathbb{Z}}$, let $L_m y(x) \in L_p(\mathbb{R})$ be the unique cardinal spline of degree $m - 1$ interpolating to $y$ at the integers. The norm of this operator, $\|L_m\|_1 = \sup \{\|L_m y\|_{L_p(\mathbb{R})}/\|y\|_{l_1(\mathbb{Z})}\}$, is called a Lebesgue constant from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$, and it was proved that $\sup_m \|L_m\|_1 = 1$

It is proved in this paper that

$$\sup_m \left\{\frac{\|L_m y\|_{1(\mathbb{R})}}{\|y\|_{l_1(\mathbb{Z})}} + \|y\|_{1(\mathbb{Z})} + \|Hy\|_{1(\mathbb{Z})}\} \right\} \leq \left(1 + \frac{\pi}{2}\right) \left(1 + \frac{\pi}{3}\right).$$

1. Introduction

Denote the classical Lebesgue space on $\mathbb{R}$ by $L_p(\mathbb{R})$, $1 \leq p \leq \infty$, and let $\| \cdot \|_{p(\mathbb{R})}$ denote its norm.

For a natural number $m$, the space $S_{m,p}(\mathbb{R}) = \{s\}$ of cardinal splines of degree $m - 1$ is taken to consist of those functions satisfying:

(i) $s \in C^{m-2}(\mathbb{R})$,
(ii) $\|s\|_{p(\mathbb{R})} < \infty$, $1 \leq p \leq \infty$,
(iii) $s$ reduces to a polynomial of degree at most $m - 1$ on each of the intervals $[\nu + m/2, \nu + m/2 + 1], \nu \in \mathbb{Z}$.

For a sequence $y = \{y_j\}_{j \in \mathbb{Z}} \in l_p(\mathbb{Z})$, we define the space of double infinite bounded sequences with the usual norm as follows:

$$\|\{y_j\}\|_{l_p(\mathbb{Z})} = \left(\sum_{j \in \mathbb{Z}} |y_j|^p\right)^{1/p}, \quad 1 < p < \infty,$$

$$\|\{y_j\}\|_{l_1(\mathbb{Z})} := \|\{y_j\}\|_{l_1(\mathbb{Z})},$$

$$\|\{y_j\}\|_{l_{\infty}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}} \{\|y_j\|\}.$$
Schoenberg [9] proved that there is a unique element $L_m y \in S_{m,p}(\mathbb{R})$ interpolating the given data at integers, i.e.,
\begin{equation}
L_m y(j) = y_j, \quad j \in \mathbb{Z}.
\end{equation}

The operator $L_m : l_p(\mathbb{Z}) \to S_{m,p}(\mathbb{R})$ is called the cardinal spline interpolation operator of order $m$ from $l_p(\mathbb{Z})$ to $L_p(\mathbb{R})$ and its norm
\begin{equation}
\|L_m\|_p = \sup \{ \|L_m y\|_{p(\mathbb{R})} : \|y\|_{l_p(\mathbb{Z})} \leq 1 \}
\end{equation}
is referred to as the $m$th Lebesgue constant for cardinal spline interpolation. These numbers were investigated previously by many authors (see [6–8]).

**Theorem A** ([9]). Let $1 < p < \infty$. Then
\begin{equation}
\|L_m\|_p \leq C_p,
\end{equation}
where the constant $C_p$ is independent of $m$.

**Theorem B** ([9]). The norms of the $m$th order cardinal spline interpolation operators from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$ satisfy
\begin{equation}
\lim_{m \to \infty} (\|L_m\|_1 - 4/\pi^2 \log m) = (2A/\pi) + 4/\pi^2 [\log(4/\pi) + \gamma],
\end{equation}
where $\gamma$ is the Euler–Mascheroni constant and
\begin{equation}
A = \int_0^\pi t^{-1} \left( \tan \left( \frac{t}{2} \right) - \frac{2}{\pi (\pi - t)} \right) dt.
\end{equation}

From Theorem B, we know that $\sup_m \{\|L_m\|_1\} = \infty$.

Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those double infinite bounded sequences $y = \{y_j\} \in l_1(\mathbb{Z})$, such that $H y = \{H y_j\} \in l_1(\mathbb{Z})$, where
\begin{equation}
H y_j = \sum_{k \neq j} \frac{y_k}{k - j}, \quad j \in \mathbb{Z},
\end{equation}
is the discrete Hilbert transform of $y$. Thus $H^1(\mathbb{Z})$ is the subspace of $l_1(\mathbb{Z})$ consisting of those sequences $y = \{y_j\}$ for which the discrete Hilbert transform also belongs to $l_1(\mathbb{Z})$. Clearly
\begin{equation}
\|\{y_j\}\|_{H^1(\mathbb{Z})} := \|\{y_j\}\|_{l_1(\mathbb{Z})} + \|\{H y_j\}\|_{l_1(\mathbb{Z})}
\end{equation}
is a norm of $H^1(\mathbb{Z})$.

$H^1(\mathbb{Z})$ was introduced by Coifman and Weiss [3, p. 622] as an important example of the Hardy space $H^1(\mathbb{X})$, associated with a space $\mathbb{X}$ of homogeneous type, in order to extend the atomic decomposition theory for the classical Hardy spaces to a more general setting. It is well known that the Hardy space $H^1(\mathbb{R})$ is a proper closed subspace of $L_1(\mathbb{R})$, and many results in harmonic analysis and approximation theory are valid on $H^1(\mathbb{R})$ but are not correct on $L_1(\mathbb{R})$. We have found the same situation exists with respect to the Lebesgue constant of the cardinal spline interpolation operator.

Our main result is the following:

**Theorem 1.** Let $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$. Then for all $m \in \mathbb{N}$
\begin{equation}
\|L_m y\|_{L(\mathbb{R})} \leq \left( 1 + \frac{\pi}{2} \right) \left( 1 + \frac{\pi}{3} \right) \|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}.
\end{equation}
2. Interpolation operator of cardinal spline

Let \( j(x) \) be the unique integer satisfying \( j(x) - \frac{1}{2} \leq x < j(x) + \frac{1}{2} \), and let
\[
\tilde{H}y(x) = \sum' y_j (x - j)^{-1},
\]
where the sum \( \sum' \) is taken over those \( j \in \mathbb{Z} \) for which \( j \neq j(x) \), and \( \tilde{H}y \) is named the mixed Hilbert transform of the sequence \( y = \{ y_j \} \). Following some ideas of [6], we have

Lemma 1. Let \( y \in H^1(\mathbb{Z}) \). Then
\[
\| \tilde{H}y \|_{1(\mathbb{R})} \leq \frac{\pi^2}{3} \| \{ y_j \} \|_{H^1(\mathbb{Z})}.
\]

Proof. From the definition of \( j(x) \), we have \( |j(x) - x| \leq \frac{1}{2} \), and for \( j \neq j(x) \), we get
\[
|j(x) - j| \leq |j(x) - x| + |x - j| \leq \frac{1}{2} + |x - j|;
\]
therefore
\[
\frac{|j(x) - j|}{x - j} \leq 1 + \frac{1}{2 |x - j|} \leq 2.
\]
For \( j \neq j(x) \),
\[
\frac{1}{x - j} = \frac{1}{j(x) - j} + \frac{(j(x) - x)(j(x) - j)}{x - j}(j(x) - j)^{-2}.
\]
Hence
\[
\left| \sum' \frac{y_j}{x - j} \right| \leq \left| \sum' \frac{y_j}{j(x) - j} \right| + \sum' \frac{|y_j|}{|j(x) - j|^2},
\]
from which we obtain
\[
\left| \sum' \frac{y_j}{x - j} \right|_{1(\mathbb{R})} \leq \int_{\mathbb{R}} \left( \left| \sum' \frac{y_j}{j(x) - j} \right| + \sum' \frac{|y_j|}{|j(x) - j|^2} \right) dx
\]
\[
= \sum_{k \in \mathbb{Z}} \int_{k - \frac{1}{2}}^{k + \frac{1}{2}} \left( \left| \sum_{j \neq k} \frac{y_j}{k - j} \right| + \sum_{j \neq k} \frac{|y_j|}{|k - j|^2} \right) dx
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_{j \neq k} \frac{y_j}{k - j} + \sum_{k \in \mathbb{Z}} \frac{1}{k^2} \sum_{j \neq k} |y_j|
\]
\[
\leq \| \{ H y_j \} \|_{1(\mathbb{Z})} + \frac{1}{3} \pi^2 \| \{ y_j \} \|_{1(\mathbb{Z})}
\]
\[
\leq \frac{1}{3} \pi^2 \| \{ y_j \} \|_{H^1(\mathbb{Z})},
\]
which completes the proof of Lemma 1. \( \square \)

Let \( W_\sigma y(x) = \sum_{k \in \mathbb{Z}} y_k \mathrm{sinc} \sigma(x - k\pi/\sigma) \), and let
\[
\| W_\sigma \|_1 = \sup \left\{ \| W_\sigma y(x) \|_{1(\mathbb{R})} : \left\| \{ (-1)^j y_j \} \right\|_{H^1(\mathbb{Z})} \leq 1 \right\},
\]
where sinc \( x := x^{-1} \sin x \) for \( x \neq 0 \) and 1 for \( x = 0 \), \( W_\sigma \) is the well-known Whittaker operator and \( W_\sigma y \) is the Whittaker cardinal series. From Lemma 1, we have
Theorem 2. Let $\sigma > 0$. Then
\begin{equation}
\|W_\sigma\|_1 \leq \left(1 + \frac{1}{3}\right)\left(\frac{\pi}{\sigma}\right).
\end{equation}

Proof. We first consider the case $\sigma = \pi$:
\begin{align*}
\|W_\pi y(x)\| &= \left|\sum_{j \in \mathbb{Z}} y_j \text{sinc} \pi(x - j)\right| \\
&\leq \frac{\sin \pi x}{\pi} \left|\sum_{j \neq \pi(x)} (-1)^j \frac{y_j}{x - j}\right| + |y_j(x)\text{sinc} \pi(x - j(x))| \\
&\leq \frac{1}{\pi} \sum_{j \neq \pi(x)} (-1)^j \frac{y_j}{x - j} + |y_j(x)|.
\end{align*}

Therefore, it follows from Lemma 1 that we have
\begin{equation}
\|W_\pi y(x)\|_{L(\mathbb{R})} \leq \pi \leq \pi \left\|\{(-1)^j y_j\}\right\|_{H^1(\mathbb{Z})} + \|\{y_j\}\|_{l(\mathbb{Z})}
\end{equation}
\begin{equation}
\leq \left(1 + \frac{\pi}{3}\right)\|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}.
\end{equation}

By changing scale, we obtain from (2.7) that
\begin{equation}
\|W_\sigma\|_1 \leq \left(1 + \frac{\pi}{3}\right)\left(\frac{\pi}{\sigma}\right).
\end{equation}

Denote by $L^m_p(\mathbb{R})$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$, the subspace of $f$ in $L_p(\mathbb{R})$ for which the $(m-1)$th derivative of $f$ exists and is locally absolutely continuous on $\mathbb{R}$, and for which $\|f^{(m)}\|_{p(\mathbb{R})}$ is finite. By the Helly theorem, if $f \in L^m_p(\mathbb{R})$, then
\begin{equation}
\|\{f(j)\}\|_{p(\mathbb{Z})} \leq \|f\|_{p(\mathbb{R})} + \|f'\|_{p(\mathbb{R})} < \infty;
\end{equation}
therefore, it follows from Schoenberg \[9\] that for every $f \in L^m_p(\mathbb{R})$, there is a unique $L_m f \in S_{m,p}(\mathbb{R})$, such that $L_m f(j) = f(j)$ for all $j \in \mathbb{Z}$. Moreover, we have

Lemma 2. Let $f \in L^m_p(\mathbb{R})$, $m \in \mathbb{N}$, and let $L_m f$ be the unique cardinal spline of degree $m-1$ interpolating to $\{f(j)\}_{j \in \mathbb{Z}}$ at the integers. Then
\begin{equation}
\|f - L_m f\|_{1(\mathbb{R})} \leq \frac{K_m}{\pi^m} \|f^{(m)}\|_{1(\mathbb{R})},
\end{equation}
where $K_m$ is the Favard constant,
\begin{equation}
K_m := \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k + 1)^{m+1}},
\end{equation}
and
\begin{equation}
1 = K_0 < K_2 < \cdots < \frac{4}{\pi} < \cdots < K_3 < K_1 = \frac{\pi}{2}.
\end{equation}

Remark 1. de Boor and Schoenberg \[2\] proved that equation (2.8) is also valid for $m$ even and $p = \infty$.

Let $E_\sigma(\mathbb{R})$, $\sigma > 0$, be the restriction on $\mathbb{R}$ of entire functions of exponential type $\sigma$, and let
\begin{equation}
B_{\sigma,p} = E_\sigma(\mathbb{R}) \cap L_p(\mathbb{R}), \quad 1 \leq p \leq \infty, \quad B_\sigma := B_{\sigma,\infty}.
\end{equation}

It is well known that $B_{\sigma,p} \subseteq B_{\sigma,q}$, $1 \leq p < q \leq \infty$.
Lemma 3 (I [p. 211] Inequality of Bernstein’s type). Let \( f \in B_{\sigma,p} \), \( 1 \leq p \leq \infty \), \( \sigma > 0 \). Then
\[
\|f'\|_{p(\mathbb{R})} \leq \sigma \|f\|_{p(\mathbb{R})}.
\]

Lemma 4 (II). Let \( y = \{y_j\} \in l_2 \). Then there is a unique \( f \in B_{\sigma,2} \), interpolating the given data \( y = \{y_j\}_{j \in \mathbb{Z}} \) at the integers, and \( f \) is represented by
\[
f(x) = \sum_{j \in \mathbb{Z}} y_j \text{sinc} \pi(x - j), \quad \text{for all } x \in \mathbb{R},
\]
and the series \( \sum_{j \in \mathbb{Z}} y_j \text{sinc} \pi(x - j) \) converges uniformly on \( \mathbb{R} \).

Proof of Theorem 1. Let \( \{(-1)^jy_j\} \in H^1(\mathbb{Z}) \). Then \( \{y_j\} \in l_2(\mathbb{Z}) \). By Lemma 4, there exists a function \( f \in B_{\sigma,2} \) such that \( f(j) = y_j \) for all \( j \in \mathbb{Z} \), hence \( f \in B_{\sigma,1} \). It follows from Theorem 2 that \( f \in L_1(\mathbb{R}) \); therefore \( f \in B_{\pi,1} \). Using Lemma 2 and Bernstein’s inequality we get
\[
\|f - L_mf\|_{1(\mathbb{R})} \leq K_m \|f(m)\|_{1(\mathbb{R})} \leq K_m \|f\|_{1(\mathbb{R})},
\]
which together with (2.9) and Theorem 2 gives
\[
\|L_my\|_{1(\mathbb{R})} = \|Lmf\|_{1(\mathbb{R})} \leq (1 + K_m) \|f\|_{1(\mathbb{R})} \\
\leq \left(1 + \frac{\pi}{2}\right) \left(1 + \frac{\pi}{3}\right) \|\{(-1)^jy_j\}\|_{H^1(\mathbb{Z})},
\]
which completes the proof of Theorem 1.

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