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A WEAK COUNTABLE CHOICE PRINCIPLE

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Abstract. A weak choice principle is introduced that is implied by both countable choice and the law of excluded middle. This principle suffices to prove that metric independence is the same as linear independence in an arbitrary normed space over a locally compact field, and to prove the fundamental theorem of algebra.

This paper is written in the context of intuitionistic logic, that is, with no implicit appeal to the law of excluded middle. Following Bishop, we use two seemingly negative expressions in a positive sense. By a **nonempty** set we mean a set that has a member (an *inhabited* set), rather than one that cannot be empty; and for elements x and y of a metric space, the notation "$x \neq y$" stands for *apartness*: there exists a positive rational number bounding the distance $d(x, y)$ away from zero.

A nonempty subset S of a metric space is **located** if, for each point x in the space, and $\varepsilon > 0$, there exists s_0 in S such that $d(x, s_0) < d(x, s) + \varepsilon$ for all s in S. That is, the distance, $d(x, S) = \inf_{s \in S} d(x, s)$, from x to S exists.

1. Bishop's principle

Bishop [1, Lemma 7, page 177] showed that if Y is a nonempty, complete, located subset of a metric space, and $x \neq y$ for each y in Y, then x is bounded away from Y. In fact, he constructed, for any point x, a point y_0 in Y such that if $x \neq y_0$, then $d(x, Y) > 0$. In the proof, Bishop tacitly uses countable choice, possibly even dependent choice.

Bishop's construction suggests the following definition: Y is **strongly reflective** if for each x there exists y_0 in Y such that if $x \neq y_0$, then x is bounded away from Y. Then Bishop's construction shows

Bishop's principle: a nonempty, complete, located subset of a metric space is strongly reflective.

From Bishop's principle it follows that if k is a locally compact field, then any two norms on k^n are equivalent (see [3, Theorem XII.4.2]). Equivalently, metric independence and linear independence are the same in any normed space over k.

Using the law of excluded middle, it is easy to show that any nonempty closed subset of a metric space is strongly reflective: let $y_0 = x$ if x is in Y, and let y_0

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be any element of Y otherwise. So Bishop's principle follows from either countable choice or the law of excluded middle.

Here is a proof of Bishop's principle from countable choice. The proof is not essentially different from Bishop's, but the appeal to countable choice is made explicit.

Theorem 1.1 (Bishop). *Countable choice entails Bishop's principle.*

Proof. Let Y be a nonempty complete located subset, and x a point. We may assume that $d(x, Y) < 1/2$. Consider the sequence of nonempty sets

$$A_n = \{(1, y) : d(x, y) < 1/n\} \cup \{(0, 0) : d(x, Y) > 1/(n + 1)\}.$$

Countable choice produces a sequence $a_n \in A_n$, which necessarily has the property that if $a_n = (0, 0)$, then $a_{n+1} = (0, 0)$. From this construct a sequence in Y by replacing $(1, y)$ by y and $(0, 0)$ by y_n, where $a_n = (1, y_n)$ and $a_{n+1} = (0, 0)$. This sequence converges to the required point y_0 in Y. □

In this proof we constructed a Cauchy sequence converging to y_0 in order to use sequential completeness. Such a procedure often requires the full axiom of countable choice. However, if completeness is defined without appeal to sequences (the proof of Theorem 2.2 shows how this works), then Bishop's principle can be established on the basis of a very weak countable axiom of choice.

2. A weak countable choice principle

The following choice principle suffices both to derive Bishop's principle and to prove the fundamental theorem of algebra. It is implied by countable choice and by the law of excluded middle.

WCC: Given a sequence A_n of nonempty sets, at most one of which is not a singleton, then there is a choice sequence $a_n \in A_n$.

What does it mean for at most one of the A_n not to be a singleton? One possibility is that if $x, y \in A_n$ and $x', y' \in A_{n'}$ with $n \neq n'$, then either $x = y$ or $x' = y'$. We will use the (possibly) stronger condition—giving a weaker axiom—that if $n \neq n'$, then either A_n or $A_{n'}$ is a singleton.

Lemma 2.1. *Suppose WCC. If r is a real number, then there exists a binary sequence λ_n such that $r \neq 0$ if and only if $\lambda_n = 1$ for some n. In fact, if $\lambda_n = 0$, then $|r| < 1/2n$, and if $\lambda_n = 1$, then $|r| > 1/(2n + 1)$.*

Proof. Consider the sequence of nonempty sets

$$\Lambda_n = \{0 : |r| < 1/2n\} \cup \{1 : |r| > 1/(2n + 1)\}.$$

It is easily seen that if $n \neq n'$, then either Λ_n or $\Lambda_{n'}$ is a singleton. So, by WCC, there exists a sequence $\lambda_n \in \Lambda_n$. □

Clearly WCC is implied by countable choice. To derive it from the law of excluded middle, note first that if all the sets A_n are singletons, there is no problem. Otherwise, let m be the index of the nonsingleton, let a_m be an element of A_m, and for $n \neq m$ let a_n be the unique element of A_n. So WCC is classically true without any choice principle.

Theorem 2.2. *WCC entails Bishop's principle.*

Proof. Let Y be a nonempty, complete, located subset of a metric space, and x a point. We may assume that $d(x,Y) < 1$. Using Lemma 2.1, construct a binary sequence λ_n such that

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad d(x,Y) < 1/2n, \\ \lambda_n = 1 \quad &\Rightarrow \quad d(x,Y) > 1/(2n+1). \end{aligned}$$

Let

$$S_n = \{y \in Y : d(x,y) < 1/2n\}.$$

Note that S_n is nonempty if $\lambda_n = 0$. Now define $B_n = \{\infty\}$ unless $\lambda_n = 0$ and $\lambda_{n+1} = 1$, in which case take $B_n = S_n$. By WCC, there exists $b_n \in B_n$. Let

$$C_n = \begin{cases} S_n & \text{if } \lambda_n = 0, \\ \{b_m\} & \text{if } \lambda_n = 1, \text{ where } \lambda_m = 0 \text{ and } \lambda_{m+1} = 1. \end{cases}$$

The diameter of C_n is at most $1/n$, so the nested sequence (C_n) determines a point y_0 in Y that is within $1/n$ of each point in C_n. If $x \neq y_0$, then there exists n such that $d(x, y_0) > 2/n$, so $d(x, C_n) > 1/n$. Thus $\lambda_n = 1$, and therefore $d(x,Y) > 1/(2n+1)$. □

3. The fundamental theorem of algebra

Without countable choice, we must distinguish between complex numbers—numbers that can be approximated arbitrarily closely by Gaussian numbers—and those complex numbers that are limits of *sequences* of Gaussian numbers (**Cauchy complex numbers**). Among the Cauchy complex numbers $\lim a_i$ one can distinguish those that are **modulated**, that is, for which there is a sequence $N_1, N_2, \ldots$ of integers such that $|a_n - \lim a_i| \leq 1/k$ if $n \geq N_k$. This distinction is similar to that between an F-number and an FR-number in Russian constructive mathematics [2], where the latter includes a regulator of convergence (the "F" stands for "fundamental" as in "fundamental sequence").

To construct a square root of an arbitrary complex number, in the absence of the law of excluded middle, requires some sort of choice principle. The reason for this is related to the fact that the function $f(z) = z^2$ does not have a continuous inverse in any neighborhood of zero—there is no problem constructing the square root of a *nonzero* complex number. More informally, the problem is that if we want to construct a square root of a, then we have to have some method which will choose between the two distinct roots of a, if a turns out to be nonzero.

There are two versions of the fundamental theorem of algebra that don't require countable choice. Ruitenburg [5] proved it without choice when the coefficients of the polynomial are modulated Cauchy complex numbers. It can also be proved for arbitrary complex numbers, but individual roots may not be constructed—rather, the *set* of roots is approximated. We make that a little more precise.

A **multiset** of size n of complex numbers is a finite sequence $z_1, \ldots, z_n$. The **distance** between two multisets $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ is the infimum, over all permutations σ of $\{1, \ldots, n\}$, of $\sup_i |z_i - w_{\sigma i}|$. This gives a metric space $M_n(\mathbf{C})$. The elements of the completion $\widehat{M}_n(\mathbf{C})$ need not be multisets, but they are approximated by multisets. To each element μ of $\widehat{M}_n(\mathbf{C})$ there corresponds a unique monic polynomial f of degree n, and the multisets approximating μ give complete factorizations of approximations to f. In [4] it is shown that, conversely, given a monic polynomial f of degree n, there exists $\mu \in \widehat{M}_n(\mathbf{C})$ (the **spectrum** of f) to which f corresponds.

Although $\mu \in \widehat{M}_n(\mathbf{C})$ is not a set, we can compute the distance from a complex number to μ, and we can compute the diameter of μ. We say that a complex number is **an element of** μ if its distance to μ is zero, and that μ is **nonempty** if it has an element. If μ is the spectrum of a monic polynomial f, then a complex number r is in μ if and only if $f(r) = 0$. So the problem of finding a root of f is the same as showing that μ is nonempty. This can be done using WCC.

Theorem 3.1. *Suppose WCC holds and $\mu \in \widehat{M}_n(\mathbf{C})$ for $n > 0$. Then μ is nonempty.*

Proof. Clearly the theorem holds for $n = 1$. The averages of the multisets approximating μ approximate a complex number r, which we may call the average of μ. (If μ is the spectrum of a polynomial $X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$, then r is simply $-c_{n-1}/n$.) Let d be the diameter of μ. Note that if $d = 0$, then $r \in \mu$. Moreover, if $s \in \mu$, then $|s - r| \le d$. Using Lemma 2.1, construct a binary sequence λ_i such that

$$\begin{aligned} \lambda_i = 0 \quad &\Rightarrow \quad d < 1/2i, \\ \lambda_i = 1 \quad &\Rightarrow \quad d > 1/(2i+1). \end{aligned}$$

Let $A_i = \{r\}$ unless $\lambda_{i-1} = 0$ and $\lambda_i = 1$, in which case let A_i be the set of all elements of μ. The set A_i is nonempty because if $d > 0$, then μ can be partitioned into two separated pieces, each in some $\widehat{M}_m(\mathbf{C})$ for $0 < m < n$, and, by induction on degree, these pieces are nonempty. Invoking WCC (again) gives a sequence $a_i \in A_i$. Finally, if $\lambda_i = 1$, redefine a_i to be a_j, where j is the smallest index such that $\lambda_j = 1$. Then a_i is a Cauchy sequence converging to an element of μ. $\square$

References

[1] Bishop, Errett, *Foundations of constructive analysis*, McGraw-Hill, 1967. MR **36:**4930

[2] Kushner, Boris A., *Lectures on constructive mathematical analysis*, AMS Translations **60**, 1984. MR **86a:**03067

[3] Mines, Ray, Fred Richman and Wim Ruitenburg, *A course in constructive algebra*, Springer-Verlag, 1988. MR **89d:**03066

[4] Richman, Fred, The fundamental theorem of algebra: a constructive development without choice, *Pacific J. Math.* (to appear), http:// www.math.fau.edu/ Richman/ html/docs.htm

[5] Ruitenburg, Wim B. G., Constructing roots of polynomials over the complex numbers. *Computational aspects of Lie group representations and related topics* (Amsterdam, 1990), 107–128, CWI Tract, **84**, Math. Centrum, Centrum Wisk. Inform., Amsterdam, 1991. MR **92g:**03085

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