BOUNDED VARIATION IN THE MEAN

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Abstract. It is shown that the concept of bounded variation in the mean is not a meaningful generalization of ordinary bounded variation. In fact, it is a characterization of functions which differ from functions of bounded variation on a zero set.

Let $f$ be a real-valued function in $L^1$ on the circle group $T$. We define the corresponding interval function by $f(I) = f(b) - f(a)$, where $I$ denotes the interval $[a, b]$. Let $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$, and $I_{kx} = [x + t_{k-1}, x + t_k]$. If

$$V_m(f) = \sup \left\{ \int_T \sum_{k=1}^n |f(I_{kx})| \, dx \right\} < \infty,$$

where the supremum is taken over all partitions, then $f$ is said to be of bounded variation in the mean (or of bounded variation in the $L^1$ norm). We denote the class of all functions which are of bounded variation in the mean by $BVM$. This concept was introduced by Moricz and Siddiqi [MS], who investigated the convergence in the mean of the partial sums of $S[f]$, the Fourier series of $f$.

If $f$ is of bounded variation ($f \in BV$) with variation $V(f, T)$, then

$$\int_T \sum_{k=1}^n |f(I_{kx})| \, dx \leq 2\pi V(f, T),$$

and so it is clear that $BV \subseteq BVM$. Clearly this integral is invariant under an alteration of $f$ on a zero set, and so a function which differs from a $BV$ function on a zero set is in $BVM$.

A straightforward calculation shows that $BVM$ is a Banach space with norm

$$\|f\|_{BVM} = \|f\|_1 + V_m(f).$$

We shall show that bounded variation in the mean implies convergence of $S[f, x]$ to $f(x)$ for every $x$ which is a symmetric Lebesgue point, i.e., for every $f$ which satisfies the symmetric Lebesgue condition,

$$\frac{1}{h} \int_0^h [f(x + t) + f(x - t) - 2f(x)] \, dt = o(1) \quad \text{as } h \to 0$$

at $x$ and, for an $f$ which satisfies this condition uniformly on a set $E$ and is bounded on $E$, the convergence is uniform.
We use a convergence test of Waterman [W]. For odd integers $n$, let

$$T_n(x,t) = \frac{f(x+t/n) - f(x + (t+\pi)/n)}{1} + \frac{f(x + (t+2\pi)/n) - f(x + (t+3\pi)/n)}{3} + \cdots + \frac{f(x + (t+ (n-1)\pi)/n) - f(x + (t+n\pi)/n)}{n}$$

and let $Q_n(x,t)$ be obtained from $T_n(x,t)$ by substituting $-t$ and $-\pi$ for $t$ and $\pi$, respectively.

**Convergence Test (Waterman).** If $f \in L^1(T)$ satisfies the symmetric Lebesgue condition $(\ast)$ and also satisfies

$$(\ast\ast) \quad \int_0^{2\pi} |T_n(x,t) + Q_n(x,t)| dt = o(1) \quad \text{as } n \to \infty,$$

then $S[f,x]$ converges to $f(x)$. If $(\ast)$ and $(\ast\ast)$ hold uniformly on a set $E$ and $f$ is bounded on $E$, then $S[f,x]$ converges uniformly to $f(x)$ on $E$.

If $f \in BVM$, then for a positive integer $k$, and $\Sigma^o$ indicating summation over odd integers,

$$\int_0^{2\pi} |T_n(x,t)| dt = \int_0^{2\pi} \left| \sum_{i=1}^{n} \frac{f(x + (t + (i-1)\pi)/n) - f(x + (i\pi)/n)}{i} \right| dt \leq \int \left| \sum_{i=1}^{k} \cdots \right| dt + \int \left| \sum_{i=k+1}^{n} \cdots \right| dt \leq k \omega_1(f, \frac{\pi}{n}) + \frac{1}{k + 1} V_m(f).$$

The first term is $o(1)$ as $n \to \infty$ for fixed $k$ and the second can be made as small as we wish by choosing $k$ large. The corresponding integral with $Q_n$ is estimated in the same manner. The test then yields convergence of $S[f,x]$. Note that $(\ast\ast)$ holds uniformly on $T$. If $f$ is bounded on $E$ and the Lebesgue condition holds uniformly on $E$, the test yields uniform convergence on $E$. Note that if $f$ is uniformly continuous on $E$, then the Lebesgue condition holds uniformly.

The similarities of the implications for the convergence of $S[f]$ of the property $BVM$ and the property $BV$ lead us to ask, “What functions are in $BVM \setminus BV$?”

It is easy to see that even exceptionally regular $f$ which are not of bounded variation may not be in $BVM$. Let $V(f,I)$ denote the variation of $f$ on the interval $I$. Suppose $f$ is a continuous function which is in $C^1(a,2\pi]$ for every $a \in (0,2\pi)$ but $V(f,[a,2\pi]) / \infty$ as $a \not \to 0$. Given a partition of $[0,2\pi]$, let

$$K_x = \{ k : kx (\text{mod } 2\pi) \subseteq [a,2\pi]\}.$$

Then

$$\sum_{k=1}^{n} |f(I_{kx})| \geq \sum_{k \in K_x} |f(I_{kx})| = \sum_{k \in K_x} |f'(kx)| (t_k - t_{k-1}).$$
for some $\theta_{kx} \in I_{kx}$. For any given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $sup(t_k - t_{k-1}) < \delta_\varepsilon$ implies

$$\sum_{k \in K_x} |f'(\theta_{kx})(t_k - t_{k-1})| \geq \int_a^{2\pi} |f'(t)| dt - \varepsilon = V(f, [a, 2\pi]) - \varepsilon.$$  

Thus, for sufficiently fine partitions, we have

$$\frac{1}{2\pi} \int_T \sum_{k=1}^n |f(I_{kx})| dx \geq V(f, [a, 2\pi]) - \varepsilon,$$

implying that $f \notin \text{BVM}$. 

This observation leads us naturally to conjecture that BVM does not constitute a true extension of BV. The following result shows that this is indeed the case.

**Theorem.** A function $f \in \text{BVM}$ if and only if there is a function $g \in \text{BV}$ such that $f = g$ a.e.

**Proof.** Consider the integral means of a function $f \in L^1$,

$$f_h(x) = \frac{1}{h} \int_0^h f(x + t) dt , \ h > 0.$$  

Note that these means are absolutely continuous and $f_h(x) \to f(x)$ a.e. as $h \searrow 0$. We have also

$$f'_h = \frac{1}{h} (f(x + h) - f(x)) \quad \text{a.e.}$$

and, therefore,

$$V(f_h, T) = \frac{1}{h} \int_T |f(x + h) - f(x)| dx.$$  

If $f \in \text{BVM}$, then there is a $C < \infty$ such that

$$\sum_{i=0}^{n-1} \int_T \left| f(x + \frac{2\pi i}{n}) - f(x + \frac{2\pi(i+1)}{n}) \right| dx < C.$$  

We note that

$$\int_T \left| f\left(x + \frac{2\pi i}{n}\right) - f\left(x + \frac{2\pi(i+1)}{n}\right) \right| dx = \int_T \left| f(x) - f\left(x + \frac{2\pi}{n}\right) \right| dx = \frac{2\pi}{n} V(f_{2\pi/n}, T).$$

Thus (***) implies that

$$2\pi V(f_{2\pi/n}, T) < C < \infty,$$

or $\{f_{2\pi/n}\}$ is of uniformly bounded variation. Choose $x_0$ such that $f(x_0)$ is finite and $f_h(x_0) \to f(x_0)$ as $h \to 0$. Then for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f_h(x_0) - f(x_0)| < \varepsilon \quad \text{if} \quad 0 < h < \delta$$

and so

$$|f_h(x_0)| < |f(x_0)| + \varepsilon.$$
for such $h$. Thus the sequence $\{f_{2\pi/n}(x_0)\}$ is bounded. By Helly’s theorem we may deduce the existence of an increasing sequence of the positive integers $\{n_k\}$ and a function $g \in BV$ such that

$$f_{2\pi/n_k}(x) \to g(x)$$

for every $x$ as $k \to \infty$. However

$$f_{2\pi/n_k}(x) \to f(x) \quad \text{a.e.}$$

and so

$$f(x) = g(x) \in BV \quad \text{a.e.}$$

\[ \square \]

References
