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BOUNDED VARIATION IN THE MEAN

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ABSTRACT. It is shown that the concept of bounded variation in the mean is not a meaningful generalization of ordinary bounded variation. In fact, it is a characterization of functions which differ from functions of bounded variation on a zero set.

Let f be a real-valued function in L^1 on the circle group T. We define the corresponding interval function by $f(I) = f(b) - f(a)$, where I denotes the interval $[a, b]$. Let $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$, and $I_{kx} = [x + t_{k-1}, x + t_k]$. If

$$V_m(f) = \sup\{\int_T \sum_{k=1}^{n} |f(I_{kx})|\, dx\} < \infty,$$

where the supremum is taken over all partitions, then f is said to be of bounded variation in the mean (or of bounded variation in the L^1 norm). We denote the class of all functions which are of bounded variation in the mean by BVM. This concept was introduced by Móricz and Siddiqi [MS], who investigated the convergence in the mean of the partial sums of $S[f]$, the Fourier series of f.

If f is of bounded variation ($f \in BV$) with variation $V(f, T)$, then

$$\int_T \sum_{k=1}^{n} |f(I_{kx})|\, dx \le 2\pi V(f, T),$$

and so it is clear that $BV \subseteq BVM$. Clearly this integral is invariant under an alteration of f on a zero set, and so a function which differs from a BV function on a zero set is in BVM.

A straightforward calculation shows that BVM is a Banach space with norm

$$\|f\|_{BVM} = \|f\|_1 + V_m(f).$$

We shall show that bounded variation in the mean implies convergence of $S[f, x]$ to $f(x)$ for every x which is a symmetric Lebesgue point, i.e., for every f which satisfies the symmetric Lebesgue condition,

$$(*) \qquad \frac{1}{h}\int_0^h |f(x+t) + f(x-t) - 2f(x)|\, dt = o(1) \quad \text{as } h \to 0$$

at x and, for an f which satisfies this condition uniformly on a set E and is bounded on E, the convergence is uniform.

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We use a convergence test of Waterman [W]. For odd integers n , let

$$\begin{aligned} T_n(x,t) &= \frac{f(x+t/n)-f(x+(t+\pi)/n)}{1} \\ &+\frac{f(x+(t+2\pi)/n)-f(x+(t+3\pi)/n)}{3} \\ &+\cdots+\frac{f(x+(t+(n-1)\pi)/n)-f(x+(t+n\pi)/n)}{n} \end{aligned}$$

and let $Q_n(x,t)$ be obtained from $T_n(x,t)$ by substituting $-t$ and $-\pi$ for t and π, respectively.

Convergence Test (Waterman)**.** *If $f \in L^1(T)$ satisfies the symmetric Lebesgue condition $(*)$ and also satisfies*

$$(**) \qquad \int_{\pi}^{2\pi} |T_n(x,t)+Q_n(x,t)|\,dt = o(1) \quad \text{as } n\to\infty,$$

then $S[f,x]$ converges to $f(x)$. If $(*)$ and $(**)$ hold uniformly on a set E and f is bounded on E, then $S[f,x]$ converges uniformly to $f(x)$ on E.

If $f \in BVM$, then for a positive integer k, and Σ^o indicating summation over odd integers,

$$\begin{aligned} \int_{\pi}^{2\pi} |T_n(x,t)|\,dt &= \int_{\pi}^{2\pi} \left| \sum_{i=1}^{n}{}^{o} \; \frac{f(x+(t+(i-1))\pi/n)-f(x+(t+i\pi/n))}{i} \right| dt \\ &\le \int \left| \sum_{i=1}^{k}{}^{o} \cdots \right| dt + \int \left| \sum_{i=k+1}^{n}{}^{o} \cdots \right| dt \\ &\le k\omega_1(f,\frac{\pi}{n}) + \frac{1}{k+1} V_m(f). \end{aligned}$$

The first term is $o(1)$ as $n\to\infty$ for fixed k and the second can be made as small as we wish by choosing k large. The corresponding integral with Q_n is estimated in the same manner. The test then yields convergence of $S[f,x]$. Note that $(**)$ holds uniformly on T. If f is bounded on E and the Lebesgue condition holds uniformly on E, the test yields uniform convergence on E. Note that if f is uniformly continuous on E, then the Lebesgue condition holds uniformly.

The similarities of the implications for the convergence of $S[f]$ of the property BVM and the property BV lead us to ask, "What functions are in $BVM \setminus BV$?".

It is easy to see that even exceptionally regular f which are not of bounded variation may not be in BVM. Let $V(f,I)$ denote the variation of f on the interval I. Suppose f is a continuous function which is in $C^1(a,2\pi]$ for every $a\in(0,2\pi)$ but $V(f,[a,2\pi]) \nearrow \infty$ as $a \searrow 0$. Given a partition of $[0,2\pi]$, let

$$K_x = \{k : I_{kx}(\text{mod } 2\pi) \subseteq [a,2\pi]\}.$$

Then

$$\sum_{k=1}^{n} |f(I_{kx})| \ge \sum_{k\in K_x} |f(I_{kx})| = \sum_{k\in K_x} |f'(\theta_{kx})|\,(t_k - t_{k-1})$$

for some $\theta_{kx} \in I_{kx}$. For any given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $\sup(t_k - t_{k-1}) < \delta_\varepsilon$ implies

$$\sum_{k \in K_x} |f'(\theta_{kx})(t_k - t_{k-1})| \geq \int_a^{2\pi} |f'(t)|\, dt - \varepsilon = V(f, [a, 2\pi]) - \varepsilon.$$

Thus, for sufficiently fine partitions, we have

$$\frac{1}{2\pi} \int_T \sum_{k=1}^{n} |f(I_{kx})|\, dx \geq V(f, [a, 2\pi]) - \varepsilon,$$

implying that $f \notin BVM$.

This observation leads us naturally to conjecture that BVM does not constitute a true extension of BV. The following result shows that this is indeed the case.

Theorem. *A function $f \in BVM$ if and only if there is a function $g \in BV$ such that $f = g$ a.e.*

Proof. Consider the integral means of a function $f \in L^1$,

$$f_h(x) = \frac{1}{h} \int_0^h f(x+t)dt\ , \quad h > 0.$$

Note that these means are absolutely continuous and $f_h(x) \to f(x)$ a.e. as $h \searrow 0$. We have also

$$f_h' = \frac{1}{h}(f(x+h) - f(x)) \quad \text{a.e.}$$

and, therefore,

$$V(f_h, T) = \frac{1}{h} \int_T |f(x+h) - f(x)|\, dx.$$

If $f \in BVM$, then there is a $C < \infty$ such that

$$(***) \qquad \int_T \sum_{i=0}^{n-1} \left| f(x + \frac{2\pi i}{n}) - f(x + \frac{2\pi(i+1)}{n}) \right| dx < C.$$

We note that

$$\begin{aligned} \int_T \left| f\left(x + \frac{2\pi i}{n}\right) - f\left(x + \frac{2\pi(i+1)}{n}\right) \right| dx &= \int_T \left| f(x) - f\left(x + \frac{2\pi}{n}\right) \right| dx \\ &= \frac{2\pi}{n} V(f_{2\pi/n}, T). \end{aligned}$$

Thus $(***)$ implies that

$$2\pi V(f_{2\pi/n}, T) < C < \infty,$$

or $\{f_{2\pi/n}\}$ is of uniformly bounded variation. Choose x_0 such that $f(x_0)$ is finite and $f_h(x_0) \to f(x_0)$ as $h \to 0$. Then for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f_h(x_0) - f(x_0)| < \varepsilon \ \text{ if } \ 0 < h < \delta$$

and so

$$|f_h(x_0)| < |f(x_0)| + \varepsilon$$

for such h. Thus the sequence $\{f_{2\pi/n}(x_0)\}$ is bounded. By Helly's theorem we may deduce the existence of an increasing sequence of the positive integers $\{n_k\}$ and a function $g \in BV$ such that

$$f_{2\pi/n_k}(x) \to g(x)$$

for every x as $k \nearrow \infty$. However

$$f_{2\pi/n_k}(x) \to f(x) \quad \text{a.e.}$$

and so

$$f(x) = g(x) \in BV \quad \text{a.e.}$$

□

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