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GENERALIZED WATSON TRANSFORMS I: GENERAL THEORY

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ABSTRACT. This paper introduces two main concepts, called a *generalized Watson transform* and a *generalized skew-Watson transform*, which extend the notion of a Watson transform from its classical setting in one variable to higher dimensional and noncommutative situations. Several construction theorems are proved which provide necessary and sufficient conditions for an operator on a Hilbert space to be a generalized Watson transform or a generalized skew-Watson transform. Later papers in this series will treat applications of the theory to infinite-dimensional representation theory and integral operators on higher dimensional spaces.

1. INTRODUCTION

This article is the first of a series of articles in which we develop the theory of generalized Watson transforms and its applications. To give some historical background, Watson transforms were first studied in 1933 by Watson [5] and Hardy and Titchmarsh [2], and were investigated further in the books by Titchmarsh [4] and Bochner and Chandrasekharan [1]. In this classical setting a Watson transform is a self-inversive integral operator

$$(Wf)(x) = \frac{d}{dx}\int_0^\infty y^{-1}h(xy)f(y)dy \tag{1.1}$$

that is unitary in $\mathrm{L}^2(0,\infty)$. In the classical theory Watson transforms were analyzed by means of the Mellin transform applied to the function h in the kernel of the operator. Consequently, the classical methods do not readily generalize to higher dimensions.

However, crucial to the classical theory is the way in which the operator W transforms under the action of dilations $x \longmapsto ax$ of the underlying space $(0,\infty)$ by positive real numbers a. Our generalization of the classical theory of Watson transforms begins with a precise group-theoretic formulation of the relationship of the operator W to the representation by dilations of the multiplicative group of positive real numbers. Let

$$(R(a)f)(x) = a^{1/2}f(xa) \tag{1.2}$$

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and

$$(L(a)f)(x) = a^{-1/2} f(a^{-1}x) \tag{1.3}$$

for any $f \in \mathrm{L}^2(0,\infty), a \in (0,\infty)$. Then both R and L are unitary representations of the multiplicative group $\mathbb{R}^+ = (0,\infty)$ on $\mathrm{L}^2(0,\infty)$, and we see from (1.1) that

$$WR(a) = L(a)W. \tag{1.4}$$

In this way, a classical Watson transform is a special case of a unitary self-inversive intertwining operator of two unitary representations.

Thus, in our generalization of the concept of a Watson transform to higher dimensions we introduce a symmetry group and two unitary representations that are naturally associated with the operator we want to study. The generalized Watson transforms or generalized skew-Watson transforms that we define are then special kinds of intertwining operators (cf. the definitions in Section 3).

One should note that the concept of a generalized Watson transform or generalized skew-Watson transform is quite general. For example, in the most abstract setting, the Hilbert space on which the generalized Watson transform or generalized skew-Watson transform acts need not be an L^2 space, and the operator need not be given as an integral operator. But in many cases in which the generalized Watson transform or generalized skew-Watson transform does act on an L^2 space, the theory implies that the operator is an integral operator and provides an integral representation for the kernel of the operator. In short, this algebraic and operator-theoretic approach avoids the difficulty encountered in the classical setting of requiring an a priori analysis of an integral kernel that can be quite complicated, involve transcendental special functions, and even be highly singular. One should also observe that the classical Watson transforms (1.1) are special cases of the generalized Watson transforms.

This paper is primarily devoted to developing the abstract theory of generalized Watson transforms and generalized skew-Watson transforms. In particular, we prove several construction theorems (Theorems 4.1 and 6.1, and their corollaries) which give necessary and sufficient conditions for an operator on a Hilbert space to be a generalized Watson transform or a generalized skew-Watson transform. The importance of these results lies in the simplicity of the conditions, which in many cases are easy to verify. Consequently, the theory has many applications. Here, we merely illustrate the applications with several examples, some of which indicate how generalized Watson transforms apply to the construction of infinite dimensional representations of $GL(2,\mathbb{R})$. Later papers will be devoted to operators on higher rank spaces and representations of higher rank reductive Lie groups.

This paper contains results from my doctoral dissertation at the University of Vermont. Let me close this introduction by expressing my deep gratitude to my thesis advisor Professor Kenneth I. Gross and to Professor Ray A. Kunze, who introduced me to the subject of group representations and suggested I study Watson transforms. Their help and support have been invaluable.

2. Construction of isometric intertwining operators

By a *unitary representation* of a topological group G on a Hilbert space H we mean a homomorphism of G into the group of unitary operators on H that is continuous relative to the strong operator topology on H. Suppose R and L are two unitary representations of G on Hilbert spaces H_1 and H_2 respectively. We

say that a bounded linear operator $A : H_1 \longrightarrow H_2$ *intertwines* R and L (or is an *intertwining operator* for R and L) if $AR(g) = L(g)A$ for all $g \in G$.

Theorem 2.1. *Let G be a topological group and R and L unitary representations of G on Hilbert spaces H_1 and H_2 respectively, and denote the inner product on H_i by $\langle\cdot|\cdot\rangle_i$ for $i = 1, 2$. Let $\{\phi_\delta : \delta \in \Delta\}$ be a subset of H_1 and $\{\psi_\delta : \delta \in \Delta\}$ a subset of H_2, where Δ is some index set. Set*

$$\Phi^\circ = \{R(g)\phi_\delta : \delta \in \Delta, g \in G\}$$

and

$$\Psi^\circ = \{R(g)\psi_\delta : \delta \in \Delta, g \in G\},$$

and assume that Φ° spans a dense subspace of H_1. Then there exists an isometric linear operator A intertwining R and L such that

$$A\phi_\delta = \psi_\delta$$

for all $\delta \in \Delta$ if and only if

$$\langle\psi_{\delta_1}|L(g)\psi_{\delta_2}\rangle_2 \;=\; \langle\phi_{\delta_1}|R(g)\phi_{\delta_2}\rangle_1 \tag{2.1}$$

for all $g \in G$ and $\delta_1, \delta_2 \in \Delta$.

Proof of Theorem 2.1. First, we prove necessity. Assume there exists an isometric operator A from H_1 to H_2 that intertwines R and L such that for all $\delta \in \Delta$

$$A\phi_\delta = \psi_\delta.$$

Then we have

$$AR(g)\phi_\delta = L(g)A\phi_\delta = L(g)\psi_\delta.$$

Since A is an isometry, it preserves inner products. Hence, it follows that

$$\langle AR(g_1)\phi_{\delta_1}|AR(g_2)\phi_{\delta_2}\rangle_2 \;=\; \langle R(g_1)\phi_{\delta_1}|R(g_2)\phi_{\delta_2}\rangle_1$$

or

$$\langle L(g_1)\psi_{\delta_1}|L(g_2)\psi_{\delta_2}\rangle_2 \;=\; \langle R(g_1)\phi_{\delta_1}|R(g_2)\phi_{\delta_2}\rangle_1$$

or finally that

$$\langle\psi_{\delta_1}|L({g_1}^{-1}g_2)\psi_{\delta_2}\rangle_2 \;=\; \langle\phi_{\delta_1}|R({g_1}^{-1}g_2)\phi_{\delta_2}\rangle_1 \tag{2.2}$$

for all $\delta_1, \delta_2 \in \Delta$ and $g_1, g_2 \in G$. If we set ${g_1}^{-1}g_2 = g$ in (2.2), we get exactly condition (2.1).

We prove (2.1) is sufficient. Assume (2.1) is true, and let Φ denote the linear subspace of H spanned by Φ°. The key step in the proof is to show that for any finite subset Λ of Δ

$$\|\sum_{i\in\Lambda} c_i L(g_i)\psi_{\delta_i}\|_2^2 \;=\; \|\sum_{i\in\Lambda} c_i R(g_i)\phi_{\delta_i}\|_1^2 \tag{2.3}$$

where $(c_i, g_i, \phi_{\delta_i}, \psi_{\delta_i}) \in \mathbb{C} \times G \times \Phi^\circ \times \Psi^\circ$ for each $i \in \Lambda$. Actually,

$$\begin{aligned}
\|\sum_{i\in\Lambda} c_i L(g_i)\psi_{\delta_i}\|_2^2 &= \langle\sum_{i\in\Lambda} c_i L(g_i)\psi_{\delta_i}|\sum_{i\in\Lambda} c_i L(g_i)\psi_{\delta_i}\rangle_2 \\
&= \sum_{i,j\in\Lambda} c_i\bar{c_j}\langle L(g_i)\psi_{\delta_i}|L(g_j)\psi_{\delta_j}\rangle_2 \\
&= \sum_{i,j\in\Lambda} c_i\bar{c_j}\langle\psi_{\delta_i}|L({g_i}^{-1}g_j)\psi_{\delta_j}\rangle_2.
\end{aligned}$$

From (2.1), the last expression can be rewritten as

$$\begin{aligned}\sum_{i,j\in\Lambda} c_i\bar{c_j}\langle\phi_{\delta_i}|R(g_i^{-1}g_j)\phi_{\delta_j}\rangle_1 &= \sum_{i,j\in\Lambda} c_i\bar{c_j}\langle R(g_i)\phi_{\delta_i}|R(g_j)\phi_{\delta_j}\rangle_1\\ &= \langle\sum_{i\in\Lambda} c_iR(g_i)\phi_{\delta_i}|\sum_{i\in\Lambda} c_iR(g_i)\phi_{\delta_i}\rangle_1\\ &= \|\sum_{i\in\Lambda} c_iR(g_i)\phi_{\delta_i}\|_1^2.\end{aligned}$$

Thus (2.3) holds, and it follows that there is a well defined linear isometry $T:\Phi\to H_2$ such that

$$T\phi_\delta = \psi_\delta$$

and

$$TR(g)\phi_\delta = L(g)\psi_\delta$$

for all $\delta\in\Delta$ and $g\in G$. Since the subspace Φ is dense in H_1, T extends uniquely to an isometry A from H_1 into H_2.

Now we prove A intertwines R and L. From the definition of A, it follows that

$$(AR(g))\phi_\delta = (TR(g))\phi_\delta = L(g)\psi_\delta = (L(g)A)\phi_\delta$$

for any $\delta\in\Delta$; and by linearity,

$$(AR(g))f = (L(g)A)f \tag{2.4}$$

for any $f\in\Phi$. Therefore, by the continuity of A, R and L, (2.4) is also true for all $f\in H_1$, and we see that A intertwines R and L.

Remarks. (1) If $A: H_1 \longrightarrow H_2$ is an isometry, then the image $A(H_1)$ is a closed subspace of H_2. Thus, if A is surjective, then A is isometric from H_1 onto H_2, or in other words, A is unitary. In particular, if in the above theorem Ψ° spans a dense subspace of H_2, then the isometry A is unitary.

(2) Suppose $H_1 = H_2 = H$ and $A: H \longrightarrow H$ is an isometry. If in addition A is self-adjoint (i.e. $A^* = A$), then A is unitary and $A^2 = I$. Similarly, if A is skew-adjoint (i.e. $A^* = -A$), then A is unitary and $A^2 = -I$.

3. Definitions

Let G be a topological group, R and L two unitary representations of G on a Hilbert space H, I the identity operator of H, and suppose W is a unitary operator that intertwines R and L. We make two definitions:

(i) W is called a *generalized Watson transform* with respect to R and L if $W^2 = I$.

(ii) W is called a *generalized skew-Watson transform* with respect to R and L if $W^2 = -I$.

Remarks. (a) From the above definition, we see that an operator W is a generalized Watson transform with respect to R and L, if and only if the following three conditions are satisfied:

(1) W is unitary.

(2) $WR(g) = L(g)W$, for all $g\in G$.

(3) $W^2 = I$.

(b) For a generalized skew-Watson transform, we replace the third condition by $W^2 = -I$. Note that the concepts of generalized Watson transform and generalized

skew-Watson transform are not independent. That is, the map $W \longmapsto iW$ takes the collection of generalized Watson transforms to the collection of generalized skew-Watson transforms. Nonetheless, it is convenient, especially for applications in representation theory, to distinguish the two kinds of transforms.

4. The construction theorem for generalized Watson transforms

Let G be a topological group, and R and L two unitary representations of G on a Hilbert space H with inner product $\langle\cdot|\cdot\rangle$. Suppose $\{\phi_\delta : \delta \in \Delta\}$ and $\{\psi_\delta : \delta \in \Delta\}$ are two subsets of H indexed by some index set Δ. Set

$$\Phi^\circ = \{R(g)\phi_\delta : \delta \in \Delta, g \in G\}.$$

Theorem 4.1. *Let $G, H, R, L, \Delta,$ and Φ° be as above, and assume Φ° spans a dense subspace of H. Then there exists a generalized Watson transform W with respect to R and L such that*

$$W\phi_\delta = \psi_\delta$$

for all $\delta \in \Delta$ if and only if the following two conditions hold:

(i) *For all $g_1, g_2 \in G$ and $\delta_1, \delta_2 \in \Delta$,*

$$\langle L(g_1)\psi_{\delta_1}|R(g_2)\phi_{\delta_2}\rangle \;=\; \langle R(g_1)\phi_{\delta_1}|L(g_2)\psi_{\delta_2}\rangle. \tag{4.1}$$

(ii) *For all $g \in G$ and $\delta_1, \delta_2 \in \Delta$,*

$$\langle \psi_{\delta_1}|L(g)\psi_{\delta_2}\rangle \;=\; \langle \phi_{\delta_1}|R(g)\phi_{\delta_2}\rangle. \tag{4.2}$$

Proof of Theorem 4.1. First, we prove the necessity. Assume there exists a generalized Watson transform W on H with respect to R and L such that

$$W\phi_\delta = \psi_\delta$$

for all $\delta \in \Delta$. Since W intertwines R and L,

$$WR(g)\phi_\delta = L(g)W\phi_\delta = L(g)\psi_\delta$$

for all $\delta \in \Delta$ and $g \in G$. Because W is unitary and self-inversive, it follows that W is self-adjoint. Hence,

$$\langle Wf|R(g)\phi_\delta\rangle \;=\; \langle f|WR(g)\phi_\delta\rangle \;=\; \langle f|L(g)\psi_\delta\rangle$$

for all $g \in G, \delta \in \Delta$, and $f \in H$. In particular, if we set $f = R(g_1)\phi_{\delta_1}, g = g_2$ and $\delta = \delta_2$, we see that

$$\langle WR(g_1)\phi_{\delta_1}|R(g_2)\phi_{\delta_2}\rangle \;=\; \langle R(g_1)\phi_{\delta_1}|L(g_2)\psi_{\delta_2}\rangle.$$

Therefore,

$$\langle L(g_1)\psi_{\delta_1}|R(g_2)\phi_{\delta_2}\rangle \;=\; \langle R(g_1)\phi_{\delta_1}|L(g_2)\psi_{\delta_2}\rangle$$

for all $g_1, g_2 \in G, \delta_1, \delta_2 \in \Delta$. This is exactly condition (4.1). Since W is unitary, (4.2) follows from Theorem 2.1.

We now prove (4.1) and (4.2) are sufficient conditions. Assume (4.1) and (4.2) are true, and let Φ denote the linear subspace of H spanned by Φ°. Since Φ is dense in H and (4.2) holds, we can apply Theorem 2.1 to obtain an isometry W of H that intertwines R and L and has the property that

$$W\phi_\delta = \psi_\delta$$

for all $\delta \in \Delta, g \in G$.

The proof will be complete if we show that W is self-adjoint. Let $f_1 \in \Phi$, $f_2 \in \Phi$. Then $f_1 = \sum_{i\in\Lambda_1} c_i R(g_i)\phi_{\delta_i}$, $f_2 = \sum_{i\in\Lambda_1} d_j R(h_j)\phi_{\delta_j}$ for some $(c_i, g_i, \phi_{\delta_i})$, $(d_j, h_j, \phi_{\delta_j}) \in \mathbb{C} \times G \times \Phi^\circ$, and

$$\begin{aligned}\langle Wf_1|f_2\rangle &= \langle \sum_{i\in\Lambda_1} c_i L(g_i)\psi_{\delta_i} | \sum_{j\in\Lambda_2} d_j R(h_j)\phi_{\delta_j}\rangle \\ &= \sum_{i\in\Lambda_1, j\in\Lambda_2} c_i \bar{d}_j \langle L(g_i)\psi_{\delta_i} | R(h_j)\phi_{\delta_j}\rangle \\ &= \sum_{i\in\Lambda_1, j\in\Lambda_2} c_i \bar{d}_j \langle R(g_i)\phi_{\delta_i} | L(h_j)\psi_{\delta_j}\rangle.\end{aligned}$$

Hence, by (4.2),

$$\langle Wf_1|f_2\rangle = \langle \sum_{i\in\Lambda_1} c_i R(g_i)\phi_{\delta_i} | \sum_{j\in\Lambda_2} d_j L(h_j)\psi_{\delta_j}\rangle = \langle f_1|Wf_2\rangle.$$

Since Φ is dense in H, this proves W is self-adjoint.

5. Corollaries of the construction theorem

There are a number of important special cases in which the hypotheses of the construction theorem (Theorem 4.1) become simple to verify. For example, in the case that the subset $\{\phi_\delta : \delta \in \Delta\}$ consists of only one vector, conditions (4.1) and (4.2) for a generalized Watson transform involve only two vectors. The result is as follows.

Corollary 5.1. *Let G be a topological group, H a Hilbert space with inner product $\langle .|.\rangle$, and R and L two unitary representations of G on H. Let $\phi, \psi \in H$, and assume that the linear space Φ generated by $\Phi^\circ = \{R(g)\phi : g \in G\}$ is dense in H. Then there exists a generalized Watson transform W on H with respect to R and L such that $W\phi = \psi$ if and only if the following two conditions hold:*

$$\langle L(g_1)\psi | R(g_2)\phi\rangle = \langle R(g_1)\phi | L(g_2)\psi\rangle \tag{5.1}$$

for all $g_1, g_2 \in G$; and

$$\langle \phi | R(g)\phi\rangle = \langle \psi | L(g)\psi\rangle \tag{5.2}$$

for all $g \in G$.

This corollary is a straightforward consequence of the construction theorem.

Remark. In Corollary 5.1, let $G = \mathbb{R}_+^\times$, the multiplicative group of positive real numbers, let $H = \mathrm{L}^2(0,\infty)$, and suppose the two unitary representations R and L of G on H are the adjusted dilations defined by

$$(R(a)f)(x) = a^{1/2} f(xa) \tag{5.3}$$

and

$$L(a) = R(a^{-1}) \tag{5.4}$$

for all $a \in \mathbb{R}_+^\times$ and $f \in \mathrm{L}^2(0,\infty)$. Then the classical Watson transforms defined and investigated in [1], [2], [4] and [5] are seen from our point of view to be generalized Watson transforms with respect to R and L. Thus, the classical theory is a special case of Corollary 5.1. For example, if we set $\phi = \chi_{(0,1)}$, the characteristic function

of the interval $(0,1)$, then conditions (5.1) and (5.2) reduce to the conditions that $\psi(x)$ is real-valued almost everywhere and

$$\int_0^\infty \psi(x)\overline{\psi(x/a)}dx = \min(a,1)$$

for all $a \in \mathbb{R}_+^\times$, which are conditions satisfied by a classical Watson transform (cf. [5], [4, Chapter 8] or [1, Chapter V].

Next, in the special case of Corollary 5.1 in which G is abelian, $L(g) = R(g^{-1})$ for all $g \in G$, and $\psi = \pm\phi$, the two conditions in Corollary 5.1 can be reduced to one as follows.

Corollary 5.2. *Assume G is abelian, let R be a unitary representation of G on a Hilbert space H, and set $L(g) = R(g^{-1})$ for all $g \in G$. Let $\phi \in H$ and suppose $\Phi^\circ = \{R(g)\phi : g \in G\}$ spans a dense subspace of H. Then there is a generalized Watson transform W on H with respect to R and L such that $W\phi = \phi$ (or $W\phi = -\phi$) if and only if $\langle\phi|R(g)\phi\rangle$ is real for all $g \in G$.*

Proof. When G is abelian, $L(g) = R(g^{-1})$ and $\psi = \phi$, (5.1) becomes

$$\langle R({g_1}^{-1})\phi|R(g_2)\phi\rangle \;=\; \langle R(g_1)\phi|R({g_2}^{-1})\phi\rangle$$

which is the same as

$$\langle\phi|R(g_1g_2)\phi\rangle \;=\; \langle R(g_2g_1)\phi|\phi\rangle$$

for any $g_1, g_2 \in G$; or

$$\langle\phi|R(g)\phi\rangle \;=\; \langle R(g)\phi|\phi\rangle$$

for any $g \in G$. Therefore, it follows that (5.1) is equivalent to the condition that $\langle\phi|R(g)\phi\rangle$ is real for all $g \in G$. By similar reasoning, (5.2) is also equivalent to the condition that $\langle\phi|R(g)\phi\rangle$ is real for all $g \in G$. This completes the proof when $\psi = \phi$. The proof for the case $\psi = -\phi$ is similar. □

The next special case concerns the setting of reductive Lie groups. For our purpose, we will say that a closed subgroup G of $GL(n,\mathbb{C})$ is *r*eductive if G is closed under the adjoint map $g \mapsto g^* = \bar{g}^t$.

Corollary 5.3. *Assume G is a reductive group, let R be a unitary representation of G on a Hilbert space H, and define the representation L by $L(g) = R({g^*}^{-1})$ for all $g \in G$. Let $\phi \in H$ be such that $\Phi^\circ = \{R(g)\phi : g \in G\}$ spans a dense subspace of H. Then there is a generalized Watson transform W on H with respect to R and L such that $W\phi = \phi$ (or $W\phi = -\phi$) if and only if*

$$\langle\phi|R(g)\phi\rangle \;=\; \langle R(g^*)\phi|\phi\rangle \tag{5.5}$$

for all $g \in G$.

The proof is similar to that of Corollary 5.2.

Remark. Note that in Corollaries 5.2 and 5.3, ϕ is an eigenvector of W. Since W is of order 2, the only eigenvalues are ± 1.

6. Generalized Skew-Watson transforms

The construction theorem for generalized Watson transforms and its corollaries can be carried over directly to the construction of generalized skew-Watson transforms (see Remark (b) in Section 3).

Theorem 6.1. *Let the hypotheses be the same as in Theorem 4.1. Then there is a generalized skew-Watson transform W on H with respect to R and L such that*

$$WR(g)\phi_\delta = L(g)\psi_\delta$$

for all $\delta \in \Delta$, $g \in G$ if and only if the following two conditions hold:

$$\langle L(g_1)\psi_{\delta_1}|R(g_2)\phi_{\delta_2}\rangle \;=\; -\langle R(g_1)\phi_{\delta_1}|L(g_2)\psi_{\delta_2}\rangle \tag{6.1}$$

for all $g_1, g_2 \in G$ and $\delta_1, \delta_2 \in \Delta$; and

$$\langle \psi_{\delta_1}|L(g)\psi_{\delta_2}\rangle \;=\; \langle \phi_{\delta_1}|R(g)\phi_{\delta_2}\rangle \tag{6.2}$$

for all $g \in G$ and $\delta_1, \delta_2 \in \Delta$.

Corollary 6.2. *Let the hypotheses be the same as in Corollary 5.1. Then there exists a generalized skew-Watson transform W on H with respect to R and L such that $W\phi = \psi$ if and only if the following two conditions hold:*

$$\langle L(g_1)\psi|R(g_2)\phi\rangle \;=\; -\langle R(g_1)\phi|L(g_2)\psi\rangle \tag{6.3}$$

for all $g_1, g_2 \in G$; and

$$\langle \phi|R(g)\phi\rangle \;=\; \langle \psi|L(g)\psi\rangle \tag{6.4}$$

for all $g \in G$.

In the special case of Corollary 6.2 in which G is abelian, $L(g) = R(g^{-1})$ for all $g \in G$, and $\psi = \pm i\phi$, we obtain the analogue for generalized skew-Watson transform of Corollary 5.2. Note in this regard that the only eigenvalues for a generalized skew-Watson transform are $\pm i$.

Corollary 6.3. *Let the hypotheses be the same as in Corollary 5.2. Then there is a generalized skew-Watson transform W on H with respect to R and L such that $W\phi = i\phi$ (or $W\phi = -i\phi$) if and only if $\langle \phi|R(g)\phi\rangle$ is real for all $g \in G$.*

Corollary 6.4. *Let the hypotheses be the same as in Corollary 5.3. Then there is a generalized skew-Watson transform W on H with respect to R and L such that $W\phi = i\phi$ (or $W\phi = -i\phi$) if and only if*

$$\langle \phi|R(g)\phi\rangle \;=\; \langle R(g^*)\phi|\phi\rangle \tag{6.5}$$

for all $g \in G$.

7. Examples

(1) Since a classical Watson transform is a special case of a generalized Watson transform (see the remark in Section 5), we can use the results of Section 5 or Section 6 to give new proofs of the unitarity of some important classical transforms. As an example, the classical *Fourier cosine transform* on $\mathrm{L}^2(0,\infty)$ is obtained by restricting the Fourier transform to the even functions in $\mathrm{L}^2(-\infty,\infty)$. The hypotheses of Corollary 5.2 are satisfied by taking $\phi(x) = e^{-x^2/2}$, and we conclude that the Fourier cosine transform is unitary on $\mathrm{L}^2(0,\infty)$. The *Fourier sine transform* on $\mathrm{L}^2(0,\infty)$ is obtained by restricting the Fourier transform to the odd functions

in $\mathrm{L}^2(-\infty,\infty)$. The hypotheses of Corollary 5.2 are satisfied with $\phi(x) = xe^{-x^2/2}$, and hence the Fourier sine transform is unitary on $\mathrm{L}^2(0,\infty)$. Since the Fourier transform is the sum of the cosine transform and i times the sine transform, we see that the *Plancherel theorem* for $\mathrm{L}^2(-\infty,\infty)$ is a simple consequence of our theory.

(2) Next take $G = \mathbb{R}_+^\times$, $H = \mathrm{L}^2(0,\infty)$, let R be given by (5.3) and set $\phi = \chi_{(0,1)}$, the characteristic function of the interval $(0,1)$. Then the hypotheses of Corollary 5.2 hold, and there exists a generalized Watson transform W such that $W\phi = \phi$ and

$$WR(a)\phi = R(a^{-1})\phi.$$

for all $a \in \mathbb{R}_+^\times$. Since W is self-adjoint,

$$\langle Wf|R(a)\phi\rangle \;=\; \langle f|R(a^{-1})\phi\rangle \tag{7.1}$$

for all $f \in \mathrm{L}^2(0,\infty)$ and $a \in \mathbb{R}_+^\times$. We write the inner product in (7.1) as an integral over $(0,\infty)$ and take derivatives with respect to a, to obtain the formula

$$(Wf)(x) = \int_0^{1/x} f(t)dt - (1/x)f(1/x) \tag{7.2}$$

when f is continuous.

(3) Let $0 < s < 1$, set $H = \mathrm{L}_s^2(\mathbb{R}) = \mathrm{L}^2(\mathbb{R}, |x|^{-s}dx)$, define the unitary representations R_s and L_s of the multiplication group $\mathbb{R}^\times$ of nonzero real numbers acting on $\mathrm{L}_s^2(\mathbb{R})$ by

$$(R_s(a))f(x) = |a|^{(1-s)/2}f(xa)$$

and

$$L_s(a) = R_s(a^{-1})$$

for $a \in \mathbb{R}^\times$ and $f \in \mathrm{L}_s^2(\mathbb{R})$, and let

$$\phi^+(x) = \int_{-\infty}^{\infty} \frac{e^{-ixy}}{(1+y^2)^{(s+1)/2}}dy.$$

Then one can show that $\phi^+ \in \mathrm{L}_s^2(\mathbb{R})$ and the dilations $\Phi_+^\circ = \{R_s(a)\phi | a \in \mathbb{R}^\times\}$ span a dense subspace Φ^+ of the space $\mathrm{L}_s^2(\mathbb{R})^+$ of even functions in $\mathrm{L}_s^2(\mathbb{R})$. By Corollary 5.2 there exists a generalized Watson transform W_s^+ on $\mathrm{L}_s^2(\mathbb{R})^+$ with respect to R_s and L_s such that $W_s^+\phi^+ = \phi^+$. Similarly, set

$$\phi^-(x) = \int_{-\infty}^{\infty} \frac{ye^{-ixy}}{(1+y^2)^{(s+3)/2}}dy.$$

The hypotheses of Corollary 5.2 again hold, and we obtain a generalized Watson transform W_s^- with respect to R_s and L_s on the space $\mathrm{L}_s^2(\mathbb{R})^-$ of odd functions in $\mathrm{L}_s^2(\mathbb{R})$ such that $W_s^-\phi^- = \phi^-$. If we take the direct sum of W_s^+ and W_s^-, we obtain a generalized Watson transform W_s on $\mathrm{L}_s^2(\mathbb{R})$ with respect to R_s and L_s.

We remark that in the next paper [6] in this series, we use this example to give a new construction of the complementary series of representations of $GL(2,\mathbb{R})$ (see the concluding remarks below), and to derive properties of classical Bessel functions as a consequence of the theory of generalized Watson transforms.

(4) Now let m be a nonnegative integer, $H = \mathrm{L}_m^2(\mathbb{R}) = \mathrm{L}^2(\mathbb{R}, |x|^{-m}dx)$, R_m the unitary representation of $\mathbb{R}^\times$ on H defined by

$$(R_m(a))f(x) = |a|^{(1-m)/2}f(xa)$$

for all $a \in \mathbb{R}^\times$ and $f \in \mathrm{L}^2_m(\mathbb{R})$, and set

$$L_m(a) = R_m(a^{-1}).$$

Define

$$\phi_m(x) = \begin{cases} x^m e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then $\phi_m \in \mathrm{L}^2_m(\mathbb{R})$ and the dilations $\Phi^\circ_m = \{R_m(a)\phi_m | a \in \mathbb{R}^\times\}$ span a dense subspace Φ_m of $\mathrm{L}^2_m(\mathbb{R})$. When m is odd, Corollary 5.2 applies to yield a generalized Watson transform W_m with respect to R_m and L_m such that

$$W_m \phi_m = (-i)^{m+1} \phi_m. \tag{7.3}$$

When m is even, Corollary 6.3 applies to yield a generalized skew-Watson transform W_m with respect to R_m and L_m such that (7.3) holds. This example leads to a new realization of the (relative) discrete series of $GL(2, \mathbb{R})$ (see the concluding remarks below).

(5) Now consider the Hankel transform of order λ, defined as the integral operator

$$(H_\lambda f)(x) = \int_0^\infty (xt)^{1/2} J_\lambda(xt) f(t) dt$$

for all functions $f \in \mathrm{L}^2(0, \infty)$ for which the integral converges absolutely, where J_λ is the Bessel functions of order $\lambda > -1$. If we take $\phi(x) = x^{\lambda+1/2} e^{-x^2/2}$ in Corollary 5.2, we see that H_λ is a generalized Watson transform with respect to the two unitary representations R and L defined in (5.3) and (5.4). In particular H_λ is unitary on $\mathrm{L}^2(0, \infty)$.

We remark that in another paper [7] in this series we generalize Example (5) to higher dimensions by developing the theory of Hankel transforms on symmetric cones from the point of view of the theory of generalized Watson transforms.

8. Concluding remarks

The theory of generalized Watson transforms and generalized skew-Watson transforms originated from our study of infinite dimensional representation theory of reductive Lie groups. To see how the theory of generalized Watson transforms applies to the construction of infinite dimensional representations, consider the general linear group $G = GL(2, \mathbb{R})$ of all non-singular 2×2 matrices $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the upper triangular (Borel) subgroup B of elements for which $c = 0$. Then B together with the *Weyl reflection* $p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ generates G. Thus, an irreducible unitary representation π of G is determined by its restriction to B and p, and the representation can be constructed explicitly from the operators $\pi(b)$ for $b \in B$, and $W = \pi(p)$. The operator W is a generalized Watson transform or a generalized skew-Watson transform with respect to R and L, where R and L are the two unitary representations of the subgroup D of diagonal matrices in G defined by

$$R\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = \pi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right)$$

and

$$L\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = R\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix}\right).$$

In the case of complementary series acting on $\mathrm{L}^2_s(\mathbb{R}), 0 < s < 1$, the operator W is given by example (3). For the (relative) discrete series acting on $\mathrm{L}^2_m(\mathbb{R})$, the operator W is given by example (4). One can also give a similar construction for the principal series. Taken collectively, all the irreducible unitary representations of $G = GL(2, \mathbb{R})$ can be constructed by the method of generalized Watson transforms. Later papers in this series will apply this method to higher rank reductive Lie groups.

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