ON FINITE $\Lambda$-SUBMODULES OF SELMER GROUPS
OF ELLIPTIC CURVES

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Abstract. In this note, we give another proof of a result of R. Greenberg on
the non-existence of non-trivial finite $\Lambda$-submodules of Selmer groups.

Let $p$ be a prime number. Let $K$ be a number field and $E$ an elliptic curve
defined over $K$. For any algebraic extension $L/K$ and any place $v$ of $L$, we denote
by $L_v$ the union of the completions at $v$ of all finite extensions of $K$ contained in
$L$. We further denote by $\mathcal{L}$ (resp. $\mathcal{L}_v$) a fixed algebraic closure of $L$ (resp. $L_v$),
and fix an immersion $\mathcal{L} \hookrightarrow \mathcal{L}_v$. Then the $p^\infty$-Selmer group of $E$ over $L$ is defined as

$$\text{Sel}_{p^\infty}(E/L) = \text{Ker}(H^1(\text{Gal}(\mathcal{L}/L), E_{p^\infty}) \to \prod_v H^1(\text{Gal}(\mathcal{L}_v/L_v), E(\mathcal{L}_v))),$$

where $E_{p^\infty}$ is the $p$-primary torsion subgroup of $E(L)$ and $v$ runs over all places of $L$.

Let $K_\infty/K$ be a $\mathbb{Z}_p$-extension and denote by $K_n$ its $n$-th layer. Put $\Gamma = \text{Gal}(K_\infty/K)$
and $\Gamma_n = \text{Gal}(K_\infty/K_n)$. Let $\Lambda$ be the completed group ring $\mathbb{Z}_p[\Lambda] = \lim_{\varphi} \mathbb{Z}_p[\Lambda/\Gamma_n]$. Then we may regard $\text{Sel}_{p^\infty}(E/K_\infty)$ as a $\Lambda$-module. Furthermore,
we may regard the Pontrjagin dual of $\text{Sel}_{p^\infty}(E/K_\infty)$,

$$\mathcal{X} := \text{Hom}_{\mathbb{Z}_p}(\text{Sel}_{p^\infty}(E/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p),$$

as a $\Lambda$-module by $(\gamma f)(x) = f(\gamma^{-1} x)$ for $f \in \mathcal{X}$ and $\gamma \in \Gamma$. Then it is known that
$\mathcal{X}$ is a finitely generated $\Lambda$-module (cf. [5, Thm. 4.5(a)]), and conjectured that $\mathcal{X}$ is
$\Lambda$-torsion under some conditions (see e.g. [3]).

We now consider finite $\Lambda$-submodules of $\mathcal{X}$ in the case where $\mathcal{X}$ is $\Lambda$-torsion. Let $X_n$ be the kernel of the restriction map

$$\text{Sel}_{p^\infty}(E/K_n) \to \text{Sel}_{p^\infty}(E/K_\infty)^{\Gamma_n}.$$ 

It is known that $X_n$ is finite and bounded as $n$ varies (cf. [5, Lem. 4.4(a)]). Hence
$X := \lim_{\varphi} X_n$ is also finite, where the projective limit is taken with respect to the
corestriction maps. Then the main theorem in this note is the following:

Theorem. Assume that $\mathcal{X}$ is $\Lambda$-torsion. Then the maximal finite $\Lambda$-submodule of
$\mathcal{X}$ is isomorphic to $X$.

By definition, if $X_n = 0$ for all sufficiently large $n$, then we have $X = 0$. In particular,
we can prove the following as a corollary of this theorem:
Corollary. Assume that $X$ is $\Lambda$-torsion and (at least) one of the following holds:

(i) $E(K)$ has no element of order $p$.

(ii) $K_\infty/K$ is the cyclotomic $\mathbb{Z}_p$-extension, and there exists a prime $v$ of $K$ above $p$ such that $E$ has good ordinary reduction or multiplicative reduction at $v$ and such that the ramification index of $v$ in $K/Q$ is less than $p - 1$.

Then $X$ has no non-zero finite $\Lambda$-submodule.

Indeed, we know $X_n = 0$ for all $n$ by the assumptions (see [5, Lem. 4.4(a)] for the case (i) and [3, Prop. 3.9] for the case (ii)). The result of this corollary is already proved by R. Greenberg ([3] Prop. 4.14, Prop. 4.15) using his result on the non-existence of finite $\Lambda$-submodules of the Pontrjagin duals of some global Galois cohomology groups ([2, Prop. 5]). We give another proof using the Cassels-Tate pairing. We remark that our method can be applied to more general contexts (as well as Greenberg’s method) using generalizations of the Cassels-Tate pairing ([1], [4]).

Proof of Theorem. Let $D_n$ be the maximal divisible subgroup of $\text{Sel}_{p^n}(E/K_n)$ and $$ C_n := \text{Sel}_{p^n}(E/K_n)/D_n. $$

Put $C_\infty := \varprojlim_n C_n$ and $D_\infty := \varprojlim_n D_n$, where the direct limits are taken with respect to the restriction maps. Denote by $X_C$ (resp. $X_D$) the Pontrjagin dual of $C_\infty$ (resp. $D_\infty$). Then we have an exact sequence of $\Lambda$-modules

$$ 0 \to X_C \to X \to X_D \to 0. $$

(1)

Since $D_\infty$ is divisible, $X_D$ has no non-zero finite $\Lambda$-submodule. Hence it suffices to prove that the maximal finite $\Lambda$-submodule of $X_C$ is isomorphic to $X$.

There exists a non-degenerate skew-symmetric Galois-equivariant pairing

$$ C_n \times C_n \to \mathbb{Q}_p/\mathbb{Z}_p $$

(cf. [2, Chap. I, §6]). Furthermore, the dual of the restriction map $C_n \to C_{n+1}$ under this pairing is the corestriction map $C_{n+1} \to C_n$. Hence we have an isomorphism of $\Lambda$-modules

$$ X_C \cong \varprojlim_n C_n, $$

(2)

where the limit is taken with respect to the corestriction maps. We have the following commutative diagram:

$$ 0 \to D_n \to \text{Sel}_{p^n}(E/K_n) \to \varphi_n \to C_n \to 0 $$

(3)

$$ 0 \to D_{\infty}^\Gamma \to \text{Sel}_{p^n}(E/K_\infty) \to C_{\infty}^\Gamma \to 0 $$

Since $X$ is $\Lambda$-torsion, the $\mathbb{Z}_p$-corank of $D_n$ is bounded as $n$ varies. Take $m$ such that $D_m$ has the largest $\mathbb{Z}_p$-corank. Then the restriction map $\psi_{m,n} : D_m \to D_n$ is surjective for $n \geq m$, since $\text{Ker}(\psi_{m,n})$ is contained in $\text{Ker}(\psi_m)$ and this is finite. Furthermore, $\text{Ker}(\psi_{m,n}) = \text{Ker}(\psi_m)$ for all sufficiently large $n$. Then $\psi_n$ is an isomorphism for such $n$. Therefore, by the snake lemma, the kernel of $\varphi_n$ is isomorphic to $X_n$ for all sufficiently large $n$, and we have an exact sequence

$$ 0 \to X \to \varprojlim_n C_n \to \varprojlim_n C_{\infty}^\Gamma $$
by taking the projective limit with respect to the corestriction maps and the norm maps. Thus, by [2], it suffices to show that $\varprojlim_n C_{\infty}^\Gamma$ has no non-zero finite $A$-submodule to complete the proof. We prove this using the same argument as in the proof of [7, Prop. 13.28]. Let $F$ be a finite $A$-submodule of $\varprojlim_n C_{\infty}^\Gamma$ and take an element $f = (f_n) \in F$ annihilated by $p^k$. Then $f_{n+k}$ is contained in $C_{\infty}^\Gamma$ for all sufficiently large $n$, since $F$ is fixed by the action of $\Gamma_n$. On the other hand, $f_{n+k}$ is mapped to $f_n$ by the norm map. Hence we have

$$f_n = p^kf_{n+k} = 0$$

for all sufficiently large $n$. Thus $f$ should be equal to zero. This completes the proof.

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