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ON FINITE Λ-SUBMODULES OF SELMER GROUPS OF ELLIPTIC CURVES

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ABSTRACT. In this note, we give another proof of a result of R. Greenberg on the non-existence of non-trivial finite Λ-submodules of Selmer groups.

Let p be a prime number. Let K be a number field and E an elliptic curve defined over K. For any algebraic extension L/K and any place v of L, we denote by L_v the union of the completions at v of all finite extensions of K contained in L. We further denote by $\overline{L}$ (resp. $\overline{L}_v$) a fixed algebraic closure of L (resp. L_v), and fix an immersion $\overline{L} \hookrightarrow \overline{L}_v$. Then the p^∞-Selmer group of E over L is defined as

$$\mathrm{Sel}_{p^\infty}(E/L) = \mathrm{Ker}\big(H^1(\mathrm{Gal}(\overline{L}/L), E_{p^\infty}) \longrightarrow \prod_v H^1(\mathrm{Gal}(\overline{L}_v/L_v), E(\overline{L}_v))\big),$$

where E_{p^∞} is the p-primary torsion subgroup of $E(\overline{L})$ and v runs over all places of L.

Let K_∞/K be a $\mathbb{Z}_p$-extension and denote by K_n its n-th layer. Put $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$. Let Λ be the completed group ring $\mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma_n]$. Then we may regard $\mathrm{Sel}_{p^\infty}(E/K_\infty)$ as a Λ-module. Furthermore, we may regard the Pontrjagin dual of $\mathrm{Sel}_{p^\infty}(E/K_\infty)$,

$$\mathfrak{X} := \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(E/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p),$$

as a Λ-module by $(\gamma f)(x) = f(\gamma^{-1}x)$ for $f \in \mathfrak{X}$ and $\gamma \in \Gamma$. Then it is known that $\mathfrak{X}$ is a finitely generated Λ-module (cf. [5, Thm. 4.5(a)]), and conjectured that $\mathfrak{X}$ is Λ-torsion under some conditions (see e.g. [3]).

We now consider finite Λ-submodules of $\mathfrak{X}$ in the case where $\mathfrak{X}$ is Λ-torsion. Let X_n be the kernel of the restriction map

$$\mathrm{Sel}_{p^\infty}(E/K_n) \longrightarrow \mathrm{Sel}_{p^\infty}(E/K_\infty)^{\Gamma_n}.$$

It is known that X_n is finite and bounded as n varies (cf. [5, Lem. 4.4(a)]). Hence $X := \varprojlim_n X_n$ is also finite, where the projective limit is taken with respect to the corestriction maps. Then the main theorem in this note is the following:

Theorem. *Assume that $\mathfrak{X}$ is Λ-torsion. Then the maximal finite Λ-submodule of $\mathfrak{X}$ is isomorphic to X.*

By definition, if $X_n = 0$ for all sufficiently large n, then we have $X = 0$. In particular, we can prove the following as a corollary of this theorem:

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Corollary. *Assume that $\mathfrak{X}$ is Λ-torsion and (at least) one of the following holds*:

(i) *$E(K)$ has no element of order p.*
(ii) *K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension, and there exists a prime v of K above p such that E has good ordinary reduction or multiplicative reduction at v and such that the ramification index of v in $K/\mathbb{Q}$ is less than $p-1$.*

Then $\mathfrak{X}$ has no non-zero finite Λ-submodule.

Indeed, we know $X_n = 0$ for all n by the assumptions (see [5, Lem. 4.4(a)] for the case (i) and [3, Prop. 3.9] for the case (ii)). The result of this corollary is already proved by R. Greenberg ([3] Prop. 4.14, Prop. 4.15) using his result on the non-existence of finite Λ-submodules of the Pontrjagin duals of some global Galois cohomology groups ([2, Prop. 5]). We give another proof using the Cassels-Tate pairing. We remark that our method can be applied to more general contexts (as well as Greenberg's method) using generalizations of the Cassels-Tate pairing ([1], [4]).

Proof of Theorem. Let D_n be the maximal divisible subgroup of $\mathrm{Sel}_{p^\infty}(E/K_n)$ and

$$C_n := \mathrm{Sel}_{p^\infty}(E/K_n)/D_n.$$

Put $C_\infty := \varinjlim_n C_n$ and $D_\infty := \varinjlim_n D_n$, where the direct limits are taken with respect to the restriction maps. Denote by $\mathfrak{X}_C$ (resp. $\mathfrak{X}_D$) the Pontrjagin dual of C_∞ (resp. D_∞). Then we have an exact sequence of Λ-modules

$$0 \longrightarrow \mathfrak{X}_C \longrightarrow \mathfrak{X} \longrightarrow \mathfrak{X}_D \longrightarrow 0. \tag{1}$$

Since D_∞ is divisible, $\mathfrak{X}_D$ has no non-zero finite Λ-submodule. Hence it suffices to prove that the maximal finite Λ-submodule of $\mathfrak{X}_C$ is isomorphic to X.

There exists a non-degenerate skew-symmetric Galois-equivariant pairing

$$C_n \times C_n \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

(cf. [6, Chap. I, §6]). Furthermore, the dual of the restriction map $C_n \to C_{n+1}$ under this pairing is the corestriction map $C_{n+1} \to C_n$. Hence we have an isomorphism of Λ-modules

$$\mathfrak{X}_C \cong \varprojlim_n C_n, \tag{2}$$

where the limit is taken with respect to the corestriction maps. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_n & \longrightarrow & \mathrm{Sel}_{p^\infty}(E/K_n) & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_n} & & \downarrow & & \downarrow{\scriptstyle \varphi_n} & & \\ 0 & \longrightarrow & D_\infty^{\Gamma_n} & \longrightarrow & \mathrm{Sel}_{p^\infty}(E/K_\infty)^{\Gamma_n} & \longrightarrow & C_\infty^{\Gamma_n} & & \end{array} \tag{3}$$

Since $\mathfrak{X}$ is Λ-torsion, the $\mathbb{Z}_p$-corank of D_n is bounded as n varies. Take m such that D_m has the largest $\mathbb{Z}_p$-corank. Then the restriction map $\psi_{m,n} : D_m \to D_n$ is surjective for $n \geq m$, since $\mathrm{Ker}(\psi_{m,n})$ is contained in $\mathrm{Ker}(\psi_m)$ and this is finite. Furthermore, $\mathrm{Ker}(\psi_{m,n}) = \mathrm{Ker}(\psi_m)$ for all sufficiently large n. Then ψ_n is an isomorphism for such n. Therefore, by the snake lemma, the kernel of φ_n is isomorphic to X_n for all sufficiently large n, and we have an exact sequence

$$0 \longrightarrow X \longrightarrow \varprojlim_n C_n \longrightarrow \varprojlim_n C_\infty^{\Gamma_n}$$

by taking the projective limit with respect to the corestriction maps and the norm maps. Thus, by (2), it suffices to show that $\varprojlim_n C_\infty^{\Gamma_n}$ has no non-zero finite Λ-submodule to complete the proof. We prove this using the same argument as in the proof of [7, Prop. 13.28]. Let F be a finite Λ-submodule of $\varprojlim_n C_\infty^{\Gamma_n}$ and take an element $f = (f_n)_n \in F$ annihilated by p^k. Then f_{n+k} is contained in $C_\infty^{\Gamma_n}$ for all sufficiently large n, since F is fixed by the action of Γ_n. On the other hand, f_{n+k} is mapped to f_n by the norm map. Hence we have

$$f_n = p^k f_{n+k} = 0$$

for all sufficiently large n. Thus f should be equal to zero. This completes the proof. □

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