Abstract. Let $G = KAN$ be the Iwasawa decomposition of a complex connected semi-simple Lie group $G$. Let $P \subset G$ be a parabolic subgroup containing $AN$, and let $(P, P)$ be its commutator subgroup. In this paper, we characterize the $K$-invariant Kahler structures on $G/(P, P)$, and study the holomorphic sections of their corresponding pre-quantum line bundles.

1. Introduction

Let $K$ be a compact connected semi-simple Lie group, let $G$ be its complexification, and let $G = KAN$ be an Iwasawa decomposition. Let $T$ be the centralizer of $A$ in $K$, so that $H = TA$ is a Cartan subgroup, and $B = HN$ is a Borel subgroup of $G$. Let $P$ be a parabolic subgroup of $G$ containing $B$, and $(P, P)$ its commutator subgroup. Each $P$ determines a subgroup $A_P \subset A$ via Langlands decomposition $P = M_P A_P N_P$ (2, p. 132). It also determines a subtorus $T_P \subset T$, which makes $H_P = T_P A_P$ complex. Since $H_P$ normalizes $(P, P)$, it has right action on $G/(P, P)$. In [3], we consider $K \times T_P$-invariant Kahler structures $\omega$ on $G/(P, P)$, and study the pre-quantum line bundle [8] corresponding to $\omega$. We then observe that the holomorphic sections of the pre-quantum line bundle constitute a nice multiplicity-free $K$-representation. In this paper, we show that if the $K$-invariant $\omega$ is not preserved by the right $T_P$-action, then the pre-quantum line bundle has no holomorphic section other than the zero section.

The Lie algebra of a Lie group shall always be denoted by its lowercase German letter. For instance, $\mathfrak{h}$ and $\mathfrak{t}_P$ are the Lie algebras of $H$ and $T_P$ respectively.

Consider the root system in $\mathfrak{h}^*$. By declaring $\mathfrak{n}$ to be the negative root spaces, we obtain a system of positive roots in $\mathfrak{h}^*$. Let $\Delta$ be the simple roots. There is a natural bijective correspondence between the parabolic subgroups $P$ containing $B$ and the subsets of $\Delta$. Namely, $P$ corresponds to $\Delta_P \subset \Delta$ by

$$\Delta_P = \{\lambda \in \Delta : (\lambda, v) \neq 0 \text{ for some } v \in \mathfrak{t}_P\}.$$  

Note that as $P$ grows bigger, $\Delta_P$ gets smaller. For example, $\Delta_B = \Delta$. 

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Proposition 3.

Fix one parabolic subgroup $P$ containing $B$, with corresponding simple roots $\Delta_P = \{\lambda_1, \ldots, \lambda_r\}$ via (1.1). Each $\lambda_i$ is integral, in the sense that there is a multiplicative homomorphism $\chi_i : H \to \mathbb{C}^\times$ such that

\begin{equation}
\chi_i(e^v) = \exp(\lambda_i, v)
\end{equation}

for all $v \in \mathfrak{h}$. Let $R_t$ denote the right action of $t \in T_P$.

**Theorem 1.** Every $K$-invariant Kähler form on $G/(P, P)$ can be uniquely expressed as

\[ \omega = \sqrt{-1} \partial \bar{\partial} F + \sum_{i=1}^{r} d\beta_i, \]

where $R_t^* \beta_i = \chi_i(t) \beta_i$ for all $t \in T_P$. So $\omega$ has a potential function if and only if it is right $T_P$-invariant, if and only if $\sum_i d\beta_i$ vanishes.

Let $\omega$ be a $K$-invariant Kähler form on $G/(P, P)$. By Theorem 1, $\omega$ is exact. Therefore it is integral, and corresponds to a pre-quantum line bundle $L$ in the sense of Kostant [8]. Namely the Chern class of $L$ is $[\omega] = 0$, and $L$ has a connection $\nabla$ whose curvature is $\omega$. A smooth section $s$ of $L$ is said to be holomorphic if $\nabla_v s = 0$ whenever $v$ is an anti-holomorphic vector field [5]. Let $H(L)$ denote the holomorphic sections of $L$. The $K$-action on $G/(P, P)$ lifts to a $K$-representation on $H(L)$. In [3], we show that if $\omega$ is right $T_P$-invariant, then every irreducible $K$-representation with highest weight in $\mathfrak{t}_P^*$ occurs exactly once in $H(L)$. The following theorem observes the opposite situation, when $\omega$ is not right $T_P$-invariant.

**Theorem 2.** Suppose that $\omega$ does not satisfy the equivalent conditions given in Theorem 1. Then $H(L) = 0$.

We remark that partial results of Theorems 1 and 2 appear in [1] and [4], for the special case where $P$ is the Borel subgroup $HN$. The present paper extends those results to general parabolic subgroups $P$.

2. Proofs of theorems

In this section, we prove the two theorems mentioned in the introduction. We start with the following topological property of $G/(P, P)$.

**Proposition 3.** $H^2(G/(P, P), \mathbb{R}) = 0$.

**Proof.** Let $K^P$ be the centralizer of $T_P$ in $K$, and $K^P_{ss} = (K^P, K^P)$ be its commutator subgroup. As a manifold, $G/(P, P) = (K/K^P_{ss}) \times A_P$ [2]. Since $A_P$ has the topology of an Euclidean space, it suffices to show that $H^2(K/K^P_{ss}, \mathbb{R}) = 0$. But $K$ is compact. So we only need to consider the DeRham subcomplex of $K$-invariant differential forms on $K/K^P_{ss}$, and show that the $H^2$ of this subcomplex vanishes. This is done via relative Lie algebra cohomology as follows.

We restrict the coadjoint representation of $K$ to $K^P_{ss}$, and get

\[ \text{Ad}^* : K^P_{ss} \to \text{Aut}(\mathfrak{t}^*). \]

We extend this representation to exterior algebras, then differentiate to get the Lie algebra representation

\[ \text{ad}^* : \mathfrak{t}^P_{ss} \to \text{End}(\wedge^q \mathfrak{t}^*). \]

The relative Lie algebra cohomology is defined by the complex

\begin{equation}
\wedge^q (\mathfrak{t}, \mathfrak{t}^P_{ss})^* = \{ \omega \in \wedge^q \mathfrak{t}^* : \iota(v) \omega = \text{ad}^*_v \omega = 0 \text{ for all } v \in \mathfrak{t}^P_{ss} \}. \end{equation}
Here $\iota(v)\omega$ denotes the interior product. We write $H^0(\mathfrak{k}, \mathfrak{k}_{ss}^P)$ for the cohomology resulting from (2.1). The elements in (2.1) can be naturally identified with $K$-invariant differential forms on $K/K_P^P$. Hence to prove the proposition, it suffices to show that

(2.2) \quad H^2(\mathfrak{k}, \mathfrak{k}_{ss}^P) = 0.

Let $\omega \in \wedge^2(\mathfrak{k}, \mathfrak{k}_{ss}^P)^*$, and suppose that $d\omega = 0$. Since $\mathfrak{k}$ is semi-simple, $H^2(\mathfrak{k}) = 0$ by the Whitehead lemma [6]. So since $\omega \in \wedge^2\mathfrak{k}^*$, there exists $\beta \in \wedge^1\mathfrak{k}^*$ such that $d\beta = \omega$. To prove (2.2), we need to show that $\beta \in \wedge^1(\mathfrak{k}, \mathfrak{k}_{ss}^P)^*$; namely

(2.3) \quad \langle \beta, v \rangle = ad^*_x\beta = 0

for all $v \in \mathfrak{k}_{ss}^P$.

Pick $v \in \mathfrak{k}_{ss}^P$. Up to linear combination, there exist $x, y \in \mathfrak{k}_{ss}^P$ such that $v = [x, y]$ because $\mathfrak{k}_{ss}^P$ is semi-simple. Then

\[
\langle \beta, v \rangle = \langle \beta, [x, y] \rangle = d\beta(x, y) = \omega(x, y) = (\iota(x)\omega)(y).
\]

Since $\omega \in \wedge^2(\mathfrak{k}, \mathfrak{k}_{ss}^P)^*$ and $x \in \mathfrak{k}_{ss}^P$, we get $\iota(x)\omega = 0$. Therefore, the last expression in (2.3) vanishes. This proves half of (2.3), and we next prove the other half of it.

Pick $x \in \mathfrak{k}_{ss}^P$ and $y \in \mathfrak{k}$. By following the arguments of (2.4), we get

\[
\langle ad^*_x\beta, y \rangle = \langle \beta, [x, y] \rangle = (\iota(x)\omega)(y) = 0.
\]

Hence $ad^*_x\beta = 0$. This completes the proof of (2.3), which implies (2.2). Proposition 3 follows.

Let $W$ be the Weyl group, acting on the roots in $\mathfrak{h}^*$. Given $\tau \in W$, we let $l(\tau)$ denote its length. Let $\rho$ denote half the sum of all positive roots.

Proof of Theorem 7. Let $\omega$ be a $K$-invariant Kähler form on $G/(P, P)$. By Proposition 3 $\omega = d\beta$ for some real 1-form $\beta$. We write

(2.5) \quad \beta = \alpha + \bar{\alpha},

where $\alpha$ is a $(0, 1)$-form. Since $\omega$ is a $(1, 1)$-form, it follows from $d\beta = \omega$ that $\bar{\partial}\alpha = \partial\bar{\alpha} = 0$. In other words, we get a Dolbeault cohomology class $[\alpha] \in H^{0,1}(G/(P, P))$. We suppress $G/(P, P)$ and write $H^{0,1}$ for simplicity.

Consider $H^{0,1}$ as a $K \times T_P$-representation space. Let $H^{0,1}_K \subset H^{0,1}$ denote the $K$-invariant cohomology classes. Since $\omega$ is $K$-invariant, averaging by $K$ if necessary, we may assume that $\beta$ and $\alpha$ of (2.5) are also $K$-invariant. In other words, $[\alpha] \in H^{0,1}_K$. For an integral weight $\lambda \in \mathfrak{t}_P^*$, let $H^{0,1}_\lambda \subset H^{0,1}$ be the cohomology classes which transform by $\lambda$ under the right $T_P$-action. By Theorem 2 of [2], $H^{0,1}_K$ splits into 1-dimensional subrepresentations $H^{0,1}_\lambda$ for all $\lambda \in \mathfrak{t}_P^*$ in which we can find $\tau \in W$ satisfying

(2.6) \quad \tau(\lambda + \rho) - \rho = 0, \quad l(\tau) = 1.

But condition (2.6) simply means that $-\lambda$ is a simple root which lies in $\mathfrak{t}_P^*$. Equivalently $-\lambda \in \Delta_P$, where $\Delta_P = \{\lambda_1, ..., \lambda_r\}$ consists of the simple roots in (1.1).
Therefore, there exist $\bar{\partial}$-closed $(0,1)$-forms $\alpha_1, \ldots, \alpha_r$ such that
\begin{equation}
[\alpha] = [\sum_{i=1}^{r} \alpha_i] \in H^0_{K1}
\end{equation}
and
\begin{equation}
[\alpha_i] \in H^0_{-\lambda_i} \subset H^0_{K1}.
\end{equation}
Here (2.7) says that
\begin{equation}
\alpha = \bar{\partial}f + \sum_{i=1}^{r} \alpha_i
\end{equation}
for some smooth function $f$. For the negative root $-\lambda_i$, the character corresponding to it via (1.2) is $\chi_{-}\lambda_i$. Therefore, (2.8) says that for all right action of $R_t$ of $t \in T_P$,
\begin{equation}
R_t^* \alpha_i = \chi_{-}\lambda_i (t^{-1}) \alpha_i = \chi_{-}(t) \alpha_i.
\end{equation}

Let $\beta_i = \alpha_i + \bar{\alpha_i}$ for all $i = 1, \ldots, r$. Then by (2.5) and (2.9),
\begin{align*}
\beta &= \alpha + \bar{\alpha} \\
&= \bar{\partial}f + \partial \bar{f} + \sum_{i=1}^{r} (\alpha_i + \bar{\alpha}_i) \\
&= \bar{\partial}f + \partial \bar{f} + \sum_{i=1}^{r} \beta_i.
\end{align*}
Therefore, by setting $F = \sqrt{-1}(\bar{f} - f)$,
\begin{align*}
\omega &= d\beta = \partial \bar{\partial}f + \bar{\partial} \partial f + \sum_{i=1}^{r} d\beta_i \\
&= \sqrt{-1} \partial \bar{\partial}F + \sum_{i=1}^{r} d\beta_i.
\end{align*}
Since $\omega, \beta$ and $\alpha$ are $K$-invariant, we can take $\beta_i, \alpha_i$ and $f$ to be $K$-invariant too. Since $f$ is a $K$-invariant function on $G/(P, P) = (K/K_{h}^*) \times A_P$, it is automatically $K \times T_P$-invariant. Therefore, $F$ and $\sqrt{-1} \partial \bar{\partial} F$ are also $K \times T_P$-invariant.

Each $\beta_i$ behaves by $\chi_{-}$ in (2.10) under the right $T_P$-action. If $\sum_{i=1}^{r} d\beta_i$ does not vanish and has a potential function, then it is right $T_P$-invariant, which is impossible. Therefore, $\omega$ has a potential function if and only if $\sum_{i=1}^{r} d\beta_i$ vanishes. This can also be seen from the nontrivial Dolbeault cohomology classes $[\alpha_i] \neq 0$. Equivalently, the vanishing of $\sum_{i=1}^{r} d\beta_i$ leaves $\omega = \sqrt{-1} \partial \bar{\partial}F$ to be right $T_P$-invariant. This proves Theorem 1.

Let $\omega$ be a $K$-invariant Kähler form on $G/(P, P)$. By Theorem 1, $\omega$ is exact, so it is in particular integral. Let $L$ be the pre-quantum line bundle corresponding to $\omega$. We now prove Theorem 2 concerning the holomorphic sections on $L$.

Proof of Theorem 2 Since $B \subset P$ and $N = (B, B) \subset (P, P)$, we have the natural fibration
\[ \pi : G/N \longrightarrow G/(P, P). \]
Suppose that $\omega$ is not invariant under the right $T_P$-action. Since $\pi$ intertwines with the $K \times T_P$-action, $\pi^* \omega$ is $K$-invariant but not right $T_P$-invariant. Although
\( \pi^*\omega \) is not Kähler, it is a closed (1,1)-form on \( G/N \). Therefore, \( \pi^*\omega \) accepts most arguments in [4], including Theorem 1 there. Namely, the only holomorphic section on the pre-quantum line bundle of \( \pi^*\omega \) is the zero section.

Let \( L \) be the pre-quantum line bundle corresponding to \( \omega \). Then \( \pi^*L \) is the pre-quantum line bundle corresponding to \( \pi^*\omega \). If \( s \) is a holomorphic section on \( L \) and \( s \neq 0 \), then \( \pi^*s \) is a holomorphic section on \( \pi^*L \) and \( \pi^*s \neq 0 \). This is a contradiction, so the only holomorphic section on \( L \) is the zero section. Hence Theorem 2 is proved.

**References**


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