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COMPACT SCHUR MULTIPLIERS

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Abstract. Compact Schur multipliers on the algebra $B(\mathcal{H})$ of all bounded linear operators on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$ will be identified as the elements of the Haagerup tensor product $c_0 \otimes^h c_0$ (the completion of $c_0 \otimes c_0$ in the Haagerup norm). Other ideals of Schur multipliers related to compact operators will also be characterized.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space with a fixed orthonormal basis $\{\epsilon_i\}_{i=1}^{\infty}$, and let $B = B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We shall identify every $\xi \in \mathcal{H}$ by the (infinite) column and every $x \in B$ by the (infinite) matrix according to the given basis and often simply write $\xi = (\xi_i)$ and $x = (x_{ij})$. For a long time it has been well known that the Schur or entrywise product $x * y = (x_{ij}y_{ij})$ is an internal operation in B (see [10]) which turns B into a semisimple commutative Banach algebra (with the usual operator norm $\|\cdot\|$ and without unity). We shall refer to this algebra as the Schur algebra and write B_* instead of B in order to stress the new product in B. Some of its Banach algebra properties including the construction of its maximal ideal space were determined by Q.F. Stout [9] in a slightly more general situation.

By a *Schur multiplier* we shall understand any map $m : B_* \to B_*$ such that $m(x * y) = m(x) * y$ for every $x, y \in B_*$. It is known that every Schur multiplier is a continuous linear operator on B_* and that $M_* = M(B_*)$, the set of all Schur multipliers, is a (maximal) commutative Banach subalgebra of $B(B_*)$, the algebra of all bounded linear operators on B_* (see [7], Theorem 1.1.1). The norm is inherited from $B(B_*)$ and will be called the multiplier (or the Schur) norm and denoted by $\|\cdot\|_m$ to distinguish it from the norm $\|\cdot\|$ in the Schur algebra B_*. Since $\|x * y\| \leq \|x\|\|y\|$ holds for every $x, y \in B_*$ (see [10]) the algebra B_* can be considered as a subalgebra in M_*, and we have $\|x\|_m \leq \|x\|$ for every $x \in B_*$. It is easy to verify that the two norms on B_* are not equivalent and that B_* is not a closed subalgebra in M_*. Thus, $\overline{B_*}^m$, the closure in the multiplier norm,

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is strictly larger than B_*. We shall use the subscript $_m$ also to distinguish the convergence in the multiplier norm from the convergence in the operator norm, e.g. $x_n \to_m x$ or $x = \lim_m x_n$ contrary to $x_n \to x$ or $x = \lim x_n$. From time to time we shall also need other types of convergence of multipliers. For instance, the convergence in the *point norm topology* on $B(B_*)$, i.e. $m_n \to_{pn} m$ if and only if $m_n(y) \to m(y)$ for every $y \in B_*$, or the convergence in the *point weak operator topology* on $B(B_*)$: $m_n \to_{pwot} m$ if and only if $\langle m_n(y)\xi, \eta\rangle \to \langle m(y)\xi, \eta\rangle$ for every $y \in B_*$ and $\xi, \eta \in \mathcal{H}$. (Note that the point norm topology is sometimes called the strong operator topology. However, we shall not use this term to avoid confusion with the usual strong operator topology on $B(\mathcal{H})$.) A convenient reference for general theory of multipliers of (commutative) Banach algebras is [7].

If e_{ij} is the usual matrix unit and $m \in M_*$ is any Schur multiplier, write $m_{ij} = m(e_{ij})_{ij}$ for every i, j. Then it is easy to deduce from the identity $m(x*y) = m(x)*y$ on B_*, that $m(e_{ij}) = m_{ij}e_{ij}$ and that $m(x) = (m_{ij}x_{ij})$ if $x = (x_{ij}) \in B_*$. This may be interpreted as the Schur product and, hence, written as $m * x$ (in the sequel we shall always write $m * x$ instead of $m(x)$). For example, in the case of the point norm convergence we have $m_n * y \to m * y$ for every $y \in B_*$. Moreover, the product in M_*, inherited from $B(B_*)$, can also be identified with the Schur product of the appropriate matrices, in accordance with the fact that B_* is a subalgebra in M_*. For elementary properties of Schur multipliers regarded as infinite matrices and for some generalisations see [1].

There is still another useful interpretation of Schur multipliers. Namely, we can see them as precisely those elements in $B(B(\mathcal{H}))$ which commute with every left or right multiplication by diagonal operators in $B(\mathcal{H})$ (in the given orthonormal basis). We make the following convention: if $\alpha = (a_i)$, $\beta = (b_j)$ are two complex sequences (columns), let us agree to denote by $\alpha \otimes \beta$ the matrix $(a_i b_j)$.

Lemma 1. *M_* is equal to the algebra of all bimodule endomorphisms from B to B regarding $B = B(\mathcal{H})$ as a Banach bimodule over the commutative algebra D of all diagonal operators in the given basis.*

Proof. The statement of the lemma means precisely that $m \in M_*$ if and only if $m(axb) = am(x)b$ for every $x \in B_*$ and $a, b \in D$, the algebra of diagonal operators in $B(\mathcal{H})$. To see this let $\alpha = (a_i)$ and $\beta = (b_j)$ be bounded sequences (columns) consisting of diagonal entries of a and b, respectively. Then, by the previous convention, we can write $axb = (a_i b_j x_{ij}) = (\alpha \otimes \beta) * x$, so that $\alpha \otimes \beta \in M_*$. The only if direction of the above assertion is now obvious since $m(axb) = m((\alpha \otimes \beta) * x) = (\alpha \otimes \beta) * m(x) = am(x)b$, while the converse follows from $m(x)_{ij}e_{ij} = e_{ii}m(x)e_{jj} = m(e_{ii}xe_{jj}) = x_{ij}m(e_{ij}) = m_{ij}x_{ij}e_{ij}$ for each pair i, j of indices. $\square$

In the sequel we are going to characterize the ideal of compact multipliers on B_*, first (in Section 2) as the multiplier norm closure of compact operators on $\mathcal{H}$, and then (in Section 3) as the Haagerup tensor product $c_0 \otimes^h c_0$. Similarly, we identify the ideal of weakly compact multipliers with the point norm closure of compact operators and later also relate it with the tensor product of c_0 and l_∞. Finally, the point weak operator closure of compact operators turns out to be the whole multiplier algebra M_* and can be identified with the normal Haagerup tensor product $l_\infty \otimes^{\sigma h} l_\infty$. It follows then from the duality theory for the Haagerup tensor product that M_* is the second dual of the ideal of all compact multipliers.

2. Compact multipliers

The ideal $K(\mathcal{H})$ of all compact operators is not only (a closed) ideal of $B(\mathcal{H})$ with the usual product of operators but also a closed Schur ideal, i.e. an ideal in the Schur algebra B_*, closed in the operator norm of B_* (see [9], Proposition 2.5). However, it is not a maximal proper ideal in B_* as we shall see later. According to our previous notation we shall denote it as K_*.

Thus, we can consider multipliers on K_*, that is, bounded linear maps $m : K_* \to K_*$ with the property $m(x*y) = m(x)*y$ for every $x, y \in K_*$. It is natural to denote the set of all multipliers on K_* by $M(K_*)$. However, both kinds of multipliers are closely related.

Lemma 2. *The second Banach adjoint of a multiplier $m \in M(K_*)$ is a Schur multiplier $m'' \in M_*$ and for every $m'' \in M_*$ its restriction $m = m''|_{K_*}$ is a multiplier on K_*. Both multipliers have the same matrix (m_{ij}).*

Proof. We use the duality between $K_* = K$ and the trace class $T = K'$ and the duality between T and $B_* = B = T'$ given by the bilinear form $(x, y) \mapsto \operatorname{tr}(xy)$ where tr denotes the operator trace on T. For $m = (m_{ij}) \in M(K_*)$ its Banach adjoint m' on T is equal to the operator with the transposed matrix: $m' = (m_{ji})$. This easily follows from the identity $\operatorname{tr}((m*x)y) = \sum_i \sum_j (m_{ij}x_{ij})y_{ji} = \sum_i \sum_j x_{ij}(m'_{ji}y_{ji}) = \operatorname{tr}(x(m'*y))$ valid for $x \in K_*$ and $y \in T$. In the same way the Banach adjoint of m' is m'' acting on B_* via the matrix (m_{ij}). Hence, m'' is a multiplier on B_* which leaves K_* invariant and $m = m''|_{K_*}$. □

By Lemma 2 we have $M(K_*) = M_*|_{K_*}$. Since K_* has an approximate identity $\{e_n\}$ where $e_n = \sum_{i,j=1}^n e_{ij} = \epsilon_{(n)} \otimes \epsilon_{(n)}$ and $\epsilon_{(n)} = (1, 1, ..., 1, 0, ...)^\top$ with the first n entries equal to 1 for every n, it follows easily from $e_n * x \to x$, $x \in K_*$, that also $(m * e_n) * x \to m * x$ for every $m \in M_*$ and hence also for every $m \in M(K_*)$. Thus, we have $M(K_*) \subset \overline{K_*}$ (the closure in the point norm topology on $B(K_*)$). As the converse inclusion is obvious we have in fact the equality $M(K_*) = \overline{K_*}$ (see also [7], Theorem 1.1.6). Note that $\overline{K_*}$ must not be confused with $\overline{K_*}^{pn}$ (the closure in the point norm topology on $B(B_*)$) which is strictly smaller and which will play a substantial role in the sequel.

Using the same argument with the approximate identity as above one can show that K_* is an ideal also in M_*, the algebra of all Schur multipliers on B_*. However, K_* is not closed in M_* in the multiplier norm as we can see by considering the sequence $m_n = x_1 \oplus x_2 \oplus ... \oplus x_n$ with $x_n = y_n/\sqrt{n}$ where y_n is the $n \times n$-matrix with all entries equal to 1. The sequence (m_n) converges in M_* but not in B_*. Moreover, its limit does not belong to K_* because it is not a bounded operator on $\mathcal{H}$ at all. The closure $\overline{K_*}^m$ is a closed ideal in M_*. The same is true for $\overline{K_*}^{pn}$, the closure in the point norm topology, or for $\overline{K_*}^{pwot}$, the closure in the point weak operator topology. We clearly have $K_* \subset \overline{K_*}^m \subset \overline{K_*}^{pn} \subset \overline{K_*}^{pwot}$. We intend to show that these closures are different and to identify them with other known classes of multipliers.

Proposition 1. *The ideal $\overline{K_*}^m$ is equal to the set of all compact Schur multipliers on the algebra B_*.*

Proof. Since K_* is the closure of matrices with finitely many non-zero entries (the left upper corners of matrices) and since the convergence in the norm $\|.\|$ implies the convergence in the norm $\|.\|_m$, every element in $\overline{K_*}^m$ is a compact operator. Conversely, we already know that $e_n * (m * x) \to m * x$ for $x \in K_*$ and $m \in M_*$. If m is a compact multiplier, then this convergence is uniform on the unit ball in K_*. Hence, we have $m \in \overline{K_*}^m$. $\square$

Proposition 2. *The ideal* $\overline{K_*}^{pwot}$ *is equal to the set* M_* *of all Schur multipliers on* B_*.

Proof. Let $e_n = \epsilon_{(n)} \otimes \epsilon_{(n)}$ with $\epsilon_{(n)} = (1, 1, ..., 1, 0, ...)^\top$ as before. Denoting $\xi_{(n)} = \epsilon_{(n)} * \xi = (\xi_1, \xi_2, ..., \xi_n, 0, ...)^\top$ and $\eta_{(n)} = \epsilon_{(n)} * \eta = (\eta_1, \eta_2, ..., \eta_n, 0, ...)^\top$ for $\xi, \eta \in \mathcal{H}$ it is clear that $\xi_{(n)} \to \xi$ and $\eta_{(n)} \to \eta$ as $n \to \infty$. Then we have for every $m \in M_*,\ x \in B_*,\ \xi, \eta \in \mathcal{H}$ the convergence

$$\begin{aligned}\langle ((e_n * m) * x)\xi, \eta\rangle &= \langle (m * x)(\epsilon_{(n)} * \xi), \epsilon_{(n)} * \eta\rangle \\ &= \langle (m * x)\xi_{(n)}, \eta_{(n)}\rangle \to \langle (m * x)\xi, \eta\rangle.\end{aligned}$$

This means that $e_n * m \to_{pwot} m$. $\square$

Proposition 3. *The ideal* $\overline{K_*}^{pn}$ *is equal to the set of all Schur multipliers which maps the algebra* B_* *into* K_*, *i.e.* $\overline{K_*}^{pn} = \{m \in M_*;\ m * x \in K_*$ *for every* $x \in B_*\}$.

Proof. If $m \in \overline{K_*}^{pn}$, then $m * x = \lim(x_n * x)$ for every $x \in B_*$ and some sequence $x_n \in K_*$. Since $x_n * x \in K_*$, also $m * x \in K_*$. Conversely, if $m * x \in K_*$ for every $x \in B_*$, then with $\{e_n\}$, the usual approximate identity in K_*, we have $m * x = \lim((m * x) * e_n) = \lim((m * e_n) * x)$ for every $x \in B_*$; hence, $m = \lim_{pn}(m * e_n) \in \overline{K_*}^{pn}$. $\square$

Corollary 1. *The ideal* $\overline{K_*}^{pn}$ *is exactly the set of all weakly compact Schur multipliers.*

Proof. By [4], VI.4.2 and VI.4.8, a Schur multiplier $m = (m|_{K_*})''$ (see Lemma 2) is weakly compact if and only if $m(B_*) \subset K_*$. By Proposition 3 this is the case if and only if $m \in \overline{K_*}^{pn}$. $\square$

Examples. The following examples are constructed to distinguish among the ideals K_*, $\overline{K_*}^m$, $\overline{K_*}^{pn}$, and $\overline{K_*}^{pwot}$ (even if the intersection with B_* is considered).

1. We have already seen that $K_* \neq \overline{K_*}^m$. The sequence $m_n = x_1 \oplus x_2 \oplus ... \oplus x_n$, with $x_n = y_n/\sqrt{n}$ where y_n is now the $n \times n$-matrix with 1 in the first column and 0 otherwise, converges in the strong operator topology on B_* and also in the multiplier norm on B_*. Hence, its limit m belongs to $B_* \cap \overline{K_*}^m$. However, it does not belong to K_* since $m^* m = x_1^* x_1 \oplus x_2^* x_2 \oplus x_3^* x_3 \oplus ...$, where * denotes the Hilbert space adjoint, is a projection with infinite-dimensional range and, hence, not in K_*. Thus, $B_* \cap \overline{K_*}^m$ is an ideal in B_*, strictly larger than K_*. It is easy to see that it is closed in the norm $\|.\|$ of the algebra B_*.

2. Let $\alpha = (1, 1/2, 1/3, ...)^\top$ and $\epsilon_{(n)} = (1, 1, ..., 1, 0, ...)^\top$ as before. Then $\alpha \otimes \epsilon_{(n)}$ is a matrix with the first n columns equal to α and zero otherwise. Denoting by a and $e_{(n)}$ the diagonal matrix with the diagonal entries from α and $\epsilon_{(n)}$, respectively, we have $(\alpha \otimes \epsilon_{(n)}) * x = axe_{(n)}$ for every $x \in B_*$. Since a is compact and since $e_{(n)}$ converges in the strong operator topology on B_* to the identity operator, $(\alpha \otimes \epsilon_{(n)}) * x = axe_{(n)}$ converges to ax in the norm of B_*. Hence, the left multiplication by a (having a matrix with all columns equal to α) belongs to $\overline{K_*}^{pn}$. On

the other hand, since all of its eigenspaces are of infinite dimension it is not compact, and, thus, not in $\overline{K_*}^m$ by Proposition 1. However, we do not know whether $B_* \cap \overline{K_*}^{pn}$ is strictly larger than $B_* \cap \overline{K_*}^m$.

3. It is possible to show that $B_* \cap \overline{K_*}^{pn} \neq B_* \cap \overline{K_*}^{pwot}$. Take for example the identity operator $i \in B_*$ which is the limit of the projections $x_n = e_{11}+e_{22}+...+e_{nn}$ in the point weak operator topology but not in the point norm topology. Moreover, i cannot belong to $\overline{K_*}^{pn}$. If it were so, then i would be the limit in the point norm topology of a sequence $y_n \in K_*$ with only left upper corners of its matrices different from zero. But, $i = i * i \neq \lim y_n * i$ because $\|i - y_n * i\| = 1$ for every n. By Proposition 2 we also know that $B_* \cap \overline{K_*}^{pwot} = B_*$.

3. Tensor products

We relate now the above ideals to various kinds of Haagerup tensor products of C^*-algebra c_0 of all complex sequences converging to zero. First of all we may see the elements of the algebraic tensor product $l_2 \otimes l_2$ as the matrix representation of finite-rank operators on $\mathcal{H}$, identifying $\sum_{k=1}^n \alpha_k \otimes \beta_k$ with the matrix $(\sum_{k=1}^n a_{ki} b_{kj})$ if $\alpha_k = (a_{ki})$ and $\beta_k = (b_{kj})$ belong to l_2. In the same way we consider also elements in $c_0 \otimes c_0$ and $l_\infty \otimes l_\infty$ as appropriate matrices.

Proposition 4. *$B_* \cap (l_\infty \otimes l_\infty) = l_2 \otimes l_2$.*

Proof. Obviously, we have $l_2 \otimes l_2 \subset B_*$ and $l_2 \otimes l_2 \subset l_\infty \otimes l_\infty$. Conversely, let $\alpha_k, \beta_k \in l_\infty$ for $k = 1, 2, ..., n$ and let $\sum_{k=1}^n \alpha_k \otimes \beta_k$ be bounded as an operator in B_*. This means that $\sum_i |\sum_{k=1}^n a_{ki} \sum_j b_{kj}\xi_j|^2 < \infty$ for every $\xi = (\xi_j) \in l_2$ $(= \mathcal{H})$. Hence, $\sum_i |\sum_{k=1}^n a_{ki}\langle \xi, \beta_k\rangle|^2 < \infty$ also for every $\xi \in l_1$ where $\langle \xi, \beta_k\rangle = \sum_j b_{kj}\xi_j$ for $k = 1, 2, ..., n$. Take $\xi \in l_1$ such that $\langle \xi, \beta_k\rangle = 0$ for each $k \neq k_0$, and $\langle \xi, \beta_{k_0}\rangle = 1$. Then $\sum_i |a_{k_0 i}|^2 < \infty$ and so $\alpha_{k_0} \in l_2$. Since $\sum_{k=1}^n \beta_k \otimes \alpha_k$ is also bounded, being the transpose of $\sum_{k=1}^n \alpha_k \otimes \beta_k$, we get similarly that $\beta_{k_0} \in l_2$ for any k_0. Thus, we have proved $\sum_{k=1}^n \alpha_k \otimes \beta_k \in l_2 \otimes l_2$. □

Corollary 2. *$K_* \cap (c_0 \otimes c_0) = l_2 \otimes l_2$.*

Proof. This follows from $l_2 \otimes l_2 \subset K_* \cap (c_0 \otimes c_0) \subset B_* \cap (l_\infty \otimes l_\infty) = l_2 \otimes l_2$. □

There is a natural map θ from the algebraic tensor product $c_0 \otimes c_0$ into the algebra of all bimodule endomorphisms of $B = B_*$ regarded as the Banach bimodule over the diagonal algebra D (hence, to the algebra M_* of all Schur multipliers on B_* by Lemma 1). Namely, let a_k and b_k be diagonal operators (matrices) with the diagonal entries a_{ki} from α_k and b_{kj} from β_k, respectively, for every k. Then θ is given by $\theta(\sum_k \alpha_k \otimes \beta_k) = \sum_k a_k x b_k$. Recall also the definition of the Haagerup norm on the algebraic tensor product of two C^*-algebras (in this case both equal to c_0):

$$\|u\|_h = \inf \|\sum_k \alpha_k \alpha_k^*\|^{1/2} \|\sum_k \beta_k^* \beta_k\|^{1/2},$$

where the infimum is taken over all representations of $u = \sum_k \alpha_k \otimes \beta_k$ as a finite sum of elementary tensors (see e.g. [2]). Then, by the injectivity of the Haagerup tensor product (see e.g. [2] or [8]) and by the isometric imbedding of $l_\infty \otimes^h l_\infty$, the completion in the Haagerup norm, into the algebra of all completely bounded bimodule endomorphisms (see e.g. [11], Theorem 4.3), the map θ is an isometry. But, further, since the algebra D, acting on $\mathcal{H}$, has a cyclic vector, it follows from

[11], Theorem 2.1, that the completely bounded norm of bimodule endomorphisms coincides with its operator norm, and, hence, θ can be extended to an isometry from $c_0 \otimes^h c_0$ (with the Haagerup norm) into M_* equipped with the usual multiplier norm. For the sake of brevity we shall consider θ as an inclusion and the elements in $c_0 \otimes c_0$ as Schur multipliers on B_*. In this way we shall simply identify $c_0 \otimes^h c_0$ with $\overline{c_0 \otimes c_0}^m \in M_*$.

By this convention we have the following tensor product characterization of the ideal of all compact Schur multipliers:

Proposition 5. $\overline{K_*}^m = c_0 \otimes^h c_0$.

Proof. Since $K_* = \overline{l_2 \otimes l_2}$ (in the operator norm), we have also $\overline{K_*}^m = \overline{l_2 \otimes l_2}^m$ (in the multiplier norm). From $l_2 \otimes l_2 \subset \overline{c_0 \otimes c_0}^m$ we get $\overline{K_*}^m = \overline{l_2 \otimes l_2}^m \subset \overline{c_0 \otimes c_0}^m = c_0 \otimes^h c_0$. On the other hand, note that $l_2 \otimes l_2$ is dense in $c_0 \otimes^h c_0$ in the Haagerup norm because l_2 is dense in c_0 in the supremum norm and because of the inequality

$$\begin{aligned}\|\alpha \otimes \beta - \alpha' \otimes \beta'\|_h &\leq \|(\alpha - \alpha') \otimes \beta\|_h + \|\alpha' \otimes (\beta - \beta')\|_h \\ &= \|\alpha' - \alpha\|\|\beta\| + \|\alpha'\|\|\beta - \beta'\|\end{aligned}$$

for $\alpha, \alpha', \beta, \beta' \in c_0$. Thus, the equality holds in the above inclusion. $\square$

Note that l_∞ is a commutative von Neumann algebra which can be represented on the Hilbert space $\mathcal{H}$ as the algebra D of all diagonal operators (in a given basis). Its commutant in B is the same: $D' = D$. Hence, the so-called normal Haagerup tensor product $l_\infty \otimes^{\sigma h} l_\infty$, defined by Effros and Kishimoto in [6], may be identified isometrically with the algebra of *all* bimodule endomorphisms of $B = B_*$ regarded as the Banach bimodule over the diagonal algebra D (see [6], Theorem 2.5). These bimodule endomorphisms are precisely Schur multipliers on B_*. Thus, Proposition 2 yields:

Proposition 6. $\overline{K_*}^{pwot} = l_\infty \otimes^{\sigma h} l_\infty$.

Corollary 3. *M_* is the second dual of $\overline{K_*}^m$.*

Proof. This follows from Propositions 2 and 6 and the fact that the normal Haagerup tensor product $l_\infty \otimes^{\sigma h} l_\infty$ coincides with the second dual of $c_0 \otimes^h c_0$ (the last assertion can be deduced for example from the general duality theory for the Haagerup tensor product in [5] and [3]). $\square$

It would be nice to express in the tensor product notation also the ideal $\overline{K_*}^{pn}$ of all weakly compact multipliers. Considering $l_\infty \otimes c_0$ and $c_0 \otimes l_\infty$ as multipliers we get the following result:

Proposition 7. $\overline{K_*}^{pn} = \overline{l_\infty \otimes c_0 + c_0 \otimes l_\infty}^{pn}$.

Proof. Note that c_0 is strongly dense in l_∞, regarded as diagonal matrices. It follows that $l_\infty \otimes c_0$ and $c_0 \otimes l_\infty$ are in the point norm closure of $c_0 \otimes c_0$ because elements in c_0 give compact diagonal operators (see also the argumentation in Example 2). Hence, $l_\infty \otimes c_0 + c_0 \otimes l_\infty \subset \overline{c_0 \otimes c_0}^{pn} \subset \overline{K_*}^{pn}$ and also $\overline{l_\infty \otimes c_0 + c_0 \otimes l_\infty}^{pn} \subset \overline{K_*}^{pn}$. On the other hand, $\overline{K_*}^{pn} = \overline{l_2 \otimes l_2}^{pn}$ and, thus, obviously, $\overline{K_*}^{pn} \subset \overline{l_\infty \otimes c_0 + c_0 \otimes l_\infty}^{pn}$. $\square$

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