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A NOTE ON TENSOR PRODUCTS OF AMPLE LINE BUNDLES ON ABELIAN VARIETIES

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ABSTRACT. Let (X, L) be a polarized abelian variety defined over the complex number field. Then we classify (X, L) with $h^0(L) \geq 2$ such that $(k+1)L$ is not k-jet ample nor k-very ample.

0. INTRODUCTION

Let X be an abelian variety defined over the complex number field, and let $L_1, \ldots, L_t$ be ample line bundles on X. Then, in [BaSz], Bauer and Szemberg proved that $L_1 + \cdots + L_{k+2}$ is k-jet ample for $t = k+2$. In particular, if (X, L) is a polarized abelian variety, then $(k+2)L$ is k-jet ample.

By the above result, it is natural and interesting to consider the following problem.

Problem. Let X be an abelian variety defined over the complex number field, and let $L_1, \ldots, L_t$ be ample line bundles on X. Then characterize $(X, L_1, \ldots, L_t)$ such that $L_1 + \cdots + L_t$ is not k-jet ample for $t \leq k+1$.

In this note, we obtain the characterization of (X, L) with $h^0(L) \geq 2$ such that $(k+1)L$ is not k-jet ample nor k-very ample.

We work over the complex number field, and we use the customary notation in algebraic geometry.

1. PRELIMINARIES

Definition 1.1. (1) (See [BeSo2].) Let (X, L) be a polarized manifold. Then for a nonnegative integer k, L is called k-jet ample if the restriction map

$$H^0(L) \to H^0(L \otimes \mathcal{O}_X/(\mathfrak{m}_{y_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{y_r}^{k_r}))$$

is surjective for any choice of distinct points $y_1, \ldots, y_r$ in X and positive integers $k_1, \ldots, k_r$ with $\sum_{i=1}^{r} k_i = k+1$, where $\mathfrak{m}_{y_i}$ denotes the maximal ideal at y_i.

(2) (See [BeSo3].) Let (X, L) be a polarized manifold. Then for a nonnegative integer k, L is called k-very ample if for any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ with

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length $\mathcal{O}_Z = k+1$, the map

$$\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_Z)$$

is surjective.

Remark 1.2. 1. If L is k-jet ample, then L is k-very ample. (See Proposition 2.2 in [BeSo2].)

2. Let (X, L) be a polarized manifold such that L is k-jet ample (resp. k-very ample). Then for any smooth projective subvariety S on X, L_S is k-jet ample (resp. k-very ample).

Theorem 1.3 (Bauer-Szemberg)**.** *Let X be an abelian variety, and let $L_1, \dots, L_{k+1}$ be ample line bundles on X. Assume that L_{k+1} has no fixed component. Then $L_1 + \cdots + L_{k+1}$ is k-jet ample for $k \geq 1$.*

Proof. See Theorem 2.1 in [BaSz]. □

Lemma 1.4 (Ohbuchi)**.** *Let X be an abelian variety with $\dim X = n \geq 2$, and let L be an ample line bundle on X. Let $|M|$ be the movable part of $|L|$, and let F be the fixed part of $|L|$. If $h^0(L) \geq 2$ and $M^n = 0$, then*

$$(X, L) \cong (A_1 \times A_2, p_1^*(D_1) + p_2^*(D_2)),$$

where A_1 and A_2 are abelian varieties, D_i is an ample divisor on A_i for $i = 1, 2$ with $h^0(D_1) = 1$, and p_i is the i-th projection $A_1 \times A_2 \to A_i$ for $i = 1, 2$.

Proof. See Lemma 4 in [O]. □

Lemma 1.5. *Let C be a smooth elliptic curve, and let D be an ample divisor on C with $\deg D \geq 2$. Then $(k+1)D$ is k-jet ample.*

Proof. We remark that for a line bundle on a smooth projective curve, the notions "k-jet ample", "k-very ample", and "k-spanned" coincide. (See Proposition 2.1 in [BeSo2].)

If $k = 0$, then D is spanned since $\deg D \geq 2$.

If $k = 1$, then D is very ample since $2 \deg D \geq 4 = 2g(C) + 2$.

If $k \geq 2$, then D is k-jet ample by Lemma 1.1 in [BeSo1]. □

Lemma 1.6. *Let X_1 and X_2 be smooth projective varieties, and let L_1 and L_2 be line bundles on X_1 and X_2 respectively. For $i = 1, 2$, assume that L_i is k_i-jet ample (resp. k_i-very ample) and let $k := \min\{k_1, k_2\}$. Then $p_1^*(L_1) + p_2^*(L_2)$ is k-jet ample (resp. k-very ample), where p_i is the i-th projection $X_1 \times X_2 \to X_i$ for $i = 1, 2$.*

Proof. See Lemma 1.8 in [BeDiSo] and Lemma 3.3 in [BeSo4]. □

2. Main theorems

Theorem 2.1. *Let (X, L) be a polarized abelian variety with $\dim X = n \geq 2$ and $h^0(L) \geq 2$. Then for $k \geq 1$, $(k+1)L$ is not k-jet ample if and only if (X, L) is the following (♣):*

(♣) *$X \cong A_1 \times \cdots \times A_m$ and $L = p_1^*(D_1) + \cdots + p_m^*(D_m)$, where each A_i is an abelian variety, D_i is an ample divisor on A_i with $h^0(D_i) = 1$ for $i = 1, \dots, m-1$ and $h^0(D_m) \geq 2$, and for some j $(\neq m)$, $(k+1)D_j$ is not k-jet ample.*

Proof. First we prove the "if" part. Assume that $(k+1)L$ is k-jet ample and (X, L) is the type (♣). By assumption, $(k+1)D_j$ is not k-jet ample. Let F_t be a fiber of the t-th projection of $A_1 \times \cdots \times A_m$. Then

$$\bigcap_{t \neq j} F_t \cong A_j$$

and

$$L_{A_j} = D_j.$$

Since $(k+1)L$ is k-jet ample, we obtain that $(k+1)L_{A_j}$ is also k-jet ample. But this is a contradiction because $L_{A_j} = D_j$.

Next we prove the "only if" part. Assume that $(k+1)L$ is not k-jet ample. By Theorem 1.3, we get $\dim \mathrm{Bs}\,|L| = n-1$. Let $|M|$ be the movable part of $|L|$, and let F be the fixed part of $|L|$.

Assume that $M^n > 0$. Then M is ample. We remark that F is nef since F is an effective divisor on an abelian variety X. Hence $L + F$ is ample. On the other hand

$$(k+1)L = (L+F) + \underbrace{L + \cdots + L}_{k-1} + M.$$

Since $\dim \mathrm{Bs}\,|M| < n-1$, we get that $(k+1)L$ is k-jet ample by Theorem 1.3. But this is a contradiction by assumption.

Hence $M^n = 0$. By Lemma 1.4, we get that $X \cong A_1 \times X_1$ and $L = p_{A,1}^*(D_1) + p_{X,1}^*(D_{X,1})$, where A_1 and X_1 are abelian varieties, $p_{A,1}$ is the projection $X \to A_1$, $p_{X,1}$ is the projection $X \to X_1$, D_1 is an ample divisor on A_1 and $D_{X,1}$ is an ample divisor on X_1 with $h^0(D_1) = 1$ and $h^0(D_{X,1}) \geq 2$.

If $(k+1)D_1$ is not k-jet ample, then this completes the proof by putting $A_2 := X_1$, $p_1 := p_{A,1}$, $p_2 := p_{X,1}$, and $D_2 := D_{X,1}$.

If $(k+1)D_1$ is k-jet ample, then $(k+1)D_{X,1}$ is not k-jet ample by Lemma 1.6. In this case, we put $p_1 := p_{A,1}$ and we go to the following step for $i = 1$, where $L_0 := L$.

Step (i). Let $L_i := L_{i-1}|_{X_i}$. Then $L_i = D_{X,i}$. Assume that $(k+1)L_i$ is not k-jet ample. Let $|M_i|$ be the movable part of $|L_i|$, and let F_i be the fixed part of $|L_i|$. Then by the same argument as above, we can prove that $F_i \neq 0$ and $M_i^{n_i} = 0$, where $n_i = \dim X_i$. Hence by Lemma 1.4, we get that $X_i \cong A_{i+1} \times X_{i+1}$ and $L_i = p_{A,i+1}^*(D_{i+1}) + p_{X,i+1}^*(D_{X,i+1})$, where A_{i+1} and X_{i+1} are abelian varieties, $p_{A,i+1}$ is the projection $X_i \to A_{i+1}$, $p_{X,i+1}$ is the projection $X_i \to X_{i+1}$, D_{i+1} is an ample divisor on A_{i+1} and $D_{X,i+1}$ is an ample divisor on X_{i+1} with $h^0(D_{i+1}) = 1$ and $h^0(D_{X,i+1}) \geq 2$.

If $(k+1)D_{i+1}$ is not k-jet ample, then this completes the proof by putting $A_{i+2} := X_{i+1}$, $p_{i+1} := p_{A,i+1} \circ p_{X,i} \circ \cdots \circ p_{X,1}$, $p_{i+2} := p_{X,i+1} \circ p_{X,i} \circ \cdots \circ p_{X,1}$, and $D_{i+2} := D_{X,i+1}$.

If $(k+1)D_{i+1}$ is k-jet ample, then $(k+1)D_{X,i+1}$ is not k-jet ample by Lemma 1.6. In this case, we put $p_{i+1} := p_{A,i+1} \circ p_{X,i} \circ \cdots \circ p_{X,1}$ and we go to Step $(i+1)$.

The above procedures come to an end after a finite number of repetitions by Lemma 1.5. This completes the proof of Theorem 2.1. □

By using Theorem 1.3 and Lemma 1.4, we can prove the following by an argument similar to that in the proof of Theorem 2.1.

Theorem 2.2. *Let X be an abelian variety, and let $L_1, \ldots, L_{k+1}$ be ample line bundles on X for $k \geq 1$ with $h^0(L_i) \geq 2$ for some i. If $L_1 + \cdots + L_{k+1}$ is not k-jet ample, then $X \cong A_1 \times A_2$, where A_1 and A_2 are abelian varieties.*

We can also prove the following theorem by the same argument as in the proof of Theorem 2.1 and Theorem 2.2.

Theorem 2.3. (1) *Let (X, L) be a polarized abelian variety with $\dim X = n \geq 2$ and $h^0(L) \geq 2$. Then for $k \geq 1$, $(k+1)L$ is not k-very ample if and only if (X, L) is the following (♣):*

(♣) *$X \cong A_1 \times \cdots \times A_m$ and $L = p_1^*(D_1) + \cdots + p_m^*(D_m)$, where each A_i is an abelian variety, D_i is an ample divisor on A_i with $h^0(D_i) = 1$ for $i = 1, \ldots, m-1$ and $h^0(D_m) \geq 2$, and for some j $(\neq m)$, $(k+1)D_j$ is not k-very ample.*

(2) *Let X be an abelian variety, and let $L_1, \ldots, L_{k+1}$ be ample line bundles on X for $k \geq 1$ with $h^0(L_i) \geq 2$ for some i. If $L_1 + \cdots + L_{k+1}$ is not k-very ample, then $X \cong A_1 \times A_2$, where A_1 and A_2 are abelian varieties.*

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