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ON REFLEXIVITY OF DIRECT SUMS

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ABSTRACT. Necessary and sufficient conditions are presented to insure that the direct sum of two reflexive representations of a finite dimensional algebra is reflexive, and it is shown that for each such algebra, there is an integer k such that the direct sum of k copies of each of its representations is reflexive. Given a ring Δ, our results are actually presented in the more general setting of Δ-representations of a ring R.

Over the last thirty years, a significant amount of attention has been given to the problem of determining when an algebra A of operators on a vector space V (often a Hilbert space) over a field K is *reflexive* in the sense that no larger algebra of operators on that space has the same lattice of invariant subspaces. (See, for example, [2], [3], [6], [7], [12], [13], [14].) Of course, A is an algebra of operators on a vector space V if and only if V is a faithful left A-module. Thus, to attack this problem from a more module theoretic point of view, based on notation of Halmos [14], the following notions were presented in [9]: For any $R-\Delta$-bimodule $M = {}_RM_\Delta$ one defines

$$\operatorname{alglat}(M) = \{\alpha \in \operatorname{End}(M_\Delta) \mid \alpha L \subseteq L \text{ for all } {}_RL \leq {}_RM\},$$

and, letting $\lambda : R \to \operatorname{alglat}(M_\Delta)$ denote the canonical ring homomorphism, M is called a *reflexive* bimodule (or Δ-representation of R) if λ is surjective. Thus the K-algebra of operators A on V is reflexive if and only if ${}_AV_K$ is a reflexive bimodule (and then we simply say that the A-module V is reflexive).

The problems we shall deal with here have their roots in the papers [5] and [6] of Deddens and Fillmore and [2] of Azoff. Stated in module theoretic terms, in the first pair of papers the question was posed and answered (affirmatively) of whether a direct sum of two finitely generated reflexive modules over an algebra generated by a single complex matrix is again reflexive; and Azoff showed that for each finite dimensional module over a $\mathbb{C}$-algebra, there is a positive integer k, depending on its dimension, such that the direct sum of k copies of that module is reflexive. Other results in this vein can be found in [4] where Brenner and Butler showed that the direct sum of two copies of the regular representation of a finite dimensional algebra is reflexive, and in Habibi and Gustafson's [12] from which the same result follows for any faithful representation of a split serial algebra. (See [9], [10] and [11] for related and more general results.)

Our main objectives are to provide, in Theorem 1.2, a necessary and sufficient condition for a direct sum of two reflexive bimodules to be reflexive; and to show, in

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Theorem 3.2, that whenever R is a left artinian ring with composition length $c({}_RR)$, there is a positive integer $k \leq c({}_RR)+1$ such that the direct sum $M^{(k)}$ of k copies of M is reflexive for any bimodule ${}_RM_\Delta$. Along the way we show how Theorem 1.2 can be employed to obtain simple proofs of some known results, and we examine a common generalization of generating and cogenerating called controlling that was introduced and employed in [9] and [10], showing in particular that if ${}_RM$ is faithful and reflexive, then $M \oplus N$ is reflexive if and only if M controls N.

1. A characterization

Unless otherwise specified, all modules under consideration will be left-R, right-Δ bimodules for a fixed pair of rings R and Δ. If R is an algebra over a field K, we shall assume that $\Delta = K$. If $\alpha \in \mathrm{End}(M_\Delta)$ and $\beta \in \mathrm{End}(N_\Delta)$, we shall write $(\alpha, \beta) \in \mathrm{End}(M \oplus N)$ for the direct sum map $(\alpha,\beta) : (m,n) \mapsto (\alpha m, \beta n)$. Thus if $\gamma \in \mathrm{alglat}(M \oplus N)$, then $\gamma = (\gamma|_M, \gamma|_N)$.

A *common subquotient* of a pair of left modules ${}_RM$ and ${}_RN$ is a module ${}_RX$ that is isomorphic to a subquotient of both ${}_RM$ and ${}_RN$. This notion together with the following lemma allows us to determine just when the direct sum of a pair of reflexive modules is reflexive.

Lemma 1.1. *If $r, s \in R$, the ordered pair $(r,s) \in$ alglat$(M \oplus N)$ if and only if $(r-s)X = 0$, for every (equivalently, every cyclic) common subquotient of ${}_RM$ and ${}_RN$.*

Proof. Note that $a \in R$ annihilates every common subquotient of M and N if and only it annihilates each of their cyclic common subquotients. According to Goursats' Lemma, W is an R-submodule of $M \oplus N$, if and only if there are submodules

$$M_2 \leq M_1 \leq M \text{ and } N_2 \leq N_1 \leq N$$

and an R-isomorphism

$$f : M_1/M_2 \to N_1/N_2$$

such that

$$W = \{(m_1, n_1) \in M_1 \oplus N_1 \,|\, f(m_1 + M_2) = n_1 + N_2\}.$$

(Given W and the orthogonal projections π_M and π_N for $M \oplus N$, one checks that $M_1 = \pi_M(W)$, $M_2 = M \cap W$, $N_1 = \pi_N(W)$, and $N_2 = N \cap W$.)

($\Rightarrow$) If $(m_1, n_1) \in W$ and $(rm_1, sn_1) \in W$, then

$$f(rm_1 + M_2) = sn_1 + N_2 = sf(m_1 + M_2)$$

so, since f is an R-isomorphism, $(r-s)(M_1/M_2) = 0$.

($\Leftarrow$) Suppose $(r-s)(M_1/M_2) = 0$, and $f : M_1/M_2 \to N_1/N_2$ is an isomorphism and $(m_1, n_1) \in W$. Then

$$f(rm_1 + M_2) = rf(m_1 + M_2) = rn_1 + N_2 = sn_1 + N_2$$

so $(rm_1, sn_1) \in W$. ☐

Now we are able to provide the promised characterization in terms of annihilators of subquotients of M and N. The left *annihilator* of a module M is $\ell_R(M) = \{r \in R \,|\, rM = 0\}$.

Theorem 1.2. *Let M and N be reflexive, and let $\{X_i \mid i \in I\}$ represent one copy of each of the (cyclic) common subquotients of ${}_RM$ and ${}_RN$. Then $M \oplus N$ is reflexive if and only if*

$$\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i).$$

Proof. ($\Leftarrow$) Let $\alpha = (\beta, \gamma) \in \text{alglat}(M \oplus N)$. Then there are $r, s \in R$ with $\beta = r$ and $\gamma = s$, and by Lemma 1.1, $r - s \in \ell_R(\bigoplus_{i \in I} X_i)$. But then by hypothesis

$$r - s = p - q \quad \text{with} \quad p \in \ell_R(M) \text{ and } q \in \ell_R(N)$$

so that, letting

$$t = r - p = s - q,$$

we have $\alpha = \lambda(t)$.

($\Rightarrow$) Let $r \in \ell_R(\bigoplus_{i \in I} X_i)$. Then by Lemma 1.1, $(r, 0) \in \text{alglat}(M \oplus N)$. Thus, assuming that $M \oplus N$ is reflexive,

$$(r, 0) = (s, s)$$

and

$$r = (r - s) + s \in \ell_R(M) + \ell_R(N).$$

Since always $\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i)$, the proof is complete. □

We note that the proof ($\Rightarrow$) above shows that if $M \oplus N$ is reflexive (regardless of reflexivity of M and N), then $\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i)$.

From the inclusions

$$0 \subseteq \ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i) \subseteq \ell_R(X_i) \subseteq R$$

we easily obtain the following two corollaries:

Corollary 1.3. *If M and N are reflexive and have a common faithful subquotient, then $M \oplus N$ is reflexive.*

Corollary 1.4. *If R is semiperfect and M and N are reflexive and have no common composition factor, then $M \oplus N$ is reflexive.*

Proof. Suppose that $e_1, ..., e_m, e_{m+1}, ..., e_n$ is a complete set of primitive idempotents such that Re_i/Je_i is not a composition factor of M for $i = 1, ..., m$, and Re_i/Je_i is not a composition factor of N for $i = m+1, ..., n$. Then $\ell_R(M) + \ell_R(N) = R$. (See [1, Section 27].) □

Regarding the Deddens-Fillmore result, the algebra generated by a matrix over a field K, being isomorphic to a proper factor of the polynomial ring $K[x]$, is an example of a split commutative uniserial algebra, i.e., a direct product of local uniserial rings. Since a module over a direct product of rings is reflexive precisely when its corresponding components are reflexive, to show that direct sums of reflexive modules are reflexive over such an algebra R, we may assume that R is a local uniserial algebra, so that the ideals of R are linearly ordered and every R-module is a direct sum of factors of R (see [1, Section 32]). In this case, $R/\ell_R(M)$ embeds in M and every subquotient of ${}_RM$ is a factor of $R/\ell_R(M)$. In the presence of these facts Theorem 1.2 yields the following corollary almost at once. We note, however, that [10, Theorem 1] is more general.

Corollary 1.5. *The direct sum of a finite number of reflexive modules over a split uniserial algebra is reflexive.*

Proof. Since we may assume that the ideals of R are linearly ordered, given modules M and N, we may also assume that $\ell_R(M) \subseteq \ell_R(N) \subseteq \ell_R(X)$ for every common subquotient X of M and N. So since $R/\ell_R(N)$ embeds in N, we see that both sides of the desired equality are equal to $\ell_R(N)$. □

Next, as further applications of this theorem, we shall see that it yields particularly nice proofs of some key results in [10].

In [10], in order to show that direct sums of reflexive modules may fail to be reflexive over a split K-algebra R with radical J whose quiver contains a triple arrow, a pair of reflexive modules was constructed with diagrams (as in [8])

$$M: \quad \begin{matrix} u & & v \\ a\downarrow & b\searrow & c\downarrow \\ x & & y \end{matrix} \quad \text{and} \quad N: \quad \begin{matrix} w \\ a,c\downarrow \\ z \end{matrix}$$

with a, b, c linearly independent elements of $J \setminus J^2$. (The diagram indicates that $au = x$, $aw = cw = z$, etc.) Here

$$\ell_R(M) + \ell_R(N) = \ell_R(M) + K(a-c) + Kb,$$

but the only common subquotients of M and N are simple modules, so

$$\ell_R(\bigoplus_{i\in I} X_i) = \ell_R(M) + J \neq \ell_R(M) + \ell_R(N).$$

Thus the theorem shows that $M \oplus N$ is not reflexive.

In the positive vein we shall employ the following lemma and Theorem 1.2 to obtain a simple proof of a key part of [10, Proposition 4].

Lemma 1.6. *Let R be a split local K-algebra of $\dim({}_KR) \leq 3$. If M is a faithful left R-module, then every proper cyclic module is isomorphic to a subquotient of M.*

Proof. (This proof is essentially contained in the proof of [10, Proposition 4].) If R is uniserial, then ${}_RR$ embeds in M, so we may assume that $J^2 = 0$ and that $\dim({}_KJ) = 2$. Let $u \in M \setminus \operatorname{Soc} M$ and suppose that $\ell_R(u) \neq 0$. Then, since $u \notin \operatorname{Soc} M$, there is a $b \in J$ with $\ell_R(u) = Kb$. But then there is a $v \in M$ with $bv \neq 0$, and there is an $a \in J$ with $Ka = \ell_R(v)$. Thus it follows that

$$J = Ka \oplus Kb.$$

Now

$$Ru \cong R/Rb \quad \text{and} \quad Rv \cong R/Ra$$

are non-isomorphic uniserial submodules of M of composition length 2. If $Ru \cap Rv = 0$, we see that $\ell_R(u+v) = 0$. Otherwise

$$Ru \cap Rv = Kau = Kbv = \operatorname{Soc}(Ru + Rv),$$

so the annihilator of any non-zero K-linear combination of u and v is properly contained in J, and we may assume that $au = bv$. Then since

$$(ka + \ell b)(k^{-1}u - \ell^{-1}v) = au - bv = 0,$$

we see that every proper cyclic R-module is a subquotient of M. □

Now we have the promised result of Fuller, Nicholson and Watters [10].

Proposition 1.7. *Let R be a split local K-algebra of* $\dim({}_K R) \leq 3$. *If M and N are reflexive R-modules, then so is $M \oplus N$.*

Proof. We may assume that $M \oplus N$ is faithful. If neither M nor N is faithful, then $J = Ka \oplus Kb$ with $aM = 0$ and $bN = 0$. But then

$$\ell_R(M) + \ell_R(N) = J = \ell_R(R/J)$$

and the desired equality holds. If M is faithful and N is not, then by Lemma 1.6, every cyclic subquotient of N is a subquotient of M, so

$$\ell_R(M) + \ell_R(N) = \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i).$$

Suppose both M and N are faithful. If R embeds in M, then every cyclic subquotient of N is a common subquotient of M and N, so

$$\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i) = \ell_R(N) = 0.$$

Otherwise, the cyclic subquotients of M and N are all proper, so by Lemma 1.6, they are all common to both modules and again

$$\ell_R(\bigoplus_{i \in I} X_i) = 0.$$

□

2. Controlling

The following the definition in [9] or [10], given bimodules ${}_R M_\Delta$ and ${}_R N_\Delta$, we say that M *controls* N in case for each pair (α, n), with $\alpha \in \mathrm{End}(N_\Delta)$ and $n \in N$, there is a set

$$C_{(\alpha,n)} = \{(m_i, n_i) \mid i \in I\} \subseteq M \times N,$$

called a *connection* for α and n, such that, if there are $r_i \in R$ with

$$r_i m_i = 0 \text{ and } \alpha n_i = r_i n_i$$

for all $i \in I$, then $\alpha n = 0$.

Controlling is a particularly useful concept. In [9] it was shown that if N is either generated or cogenerated by subquotients of M, then M controls N; and that if M is reflexive and controls N, then $M \oplus N$ is reflexive. Here we shall present an equivalent version of controlling that yields a partial converse to this last assertion.

Proposition 2.1. *The bimodule ${}_R M_\Delta$ controls ${}_R N_\Delta$ if and only if, for each $0 \neq \alpha \in End(N_\Delta)$, there is a pair*

$$P_\alpha = (m_\alpha, n_\alpha) \in M \times N$$

with $\alpha n_\alpha \notin \ell_R(m_\alpha) n_\alpha$ (i.e., for all $r \in R$, if $\alpha n_\alpha = r n_\alpha$, then $r m_\alpha \neq 0$).

Proof. ($\Rightarrow$) Let $\alpha \in \mathrm{End}(N_\Delta)$ with $\alpha n \neq 0$ and suppose that $C_{(\alpha,n)} = \{(m_i, n_i) \mid i \in I\}$ is a connection for α and n. If $\alpha n_i \in \ell_R(m_i) n_i$ for all $i \in I$, then there are r_i with $r_i m_i = 0$ and $\alpha n_i = r_i n_i$ for all $i \in I$. But $\alpha n \neq 0$, so there is a pair (m_i, n_i) with $\alpha n_i \notin \ell_R(m_i) n_i$.

($\Leftarrow$) If $\alpha n = 0$, then any $C_{(\alpha,n)} = \{(m_i, n_i) \,|\, i \in I\}$ is a connection for α and n. If $\alpha n \neq 0$, let

$$C_{(\alpha,n)} = \{P_\alpha\} = \{(m_\alpha, n_\alpha)\}.$$

Then it never occurs that $r_\alpha m_\alpha = 0$ and $\alpha n_\alpha = r_\alpha n_\alpha$, so $C_{(\alpha,n)}$ is a connection. □

It is worthy of note that when employing Proposition 2.1 to test for controlling, one only needs to consider those $0 \neq \alpha \in$ alglat(N). Indeed, if $0 \neq \alpha \in \mathrm{End}(N_\Delta) \setminus \mathrm{alglat}(N)$, then there is a n_α such that $\alpha n_\alpha \notin Rn_\alpha$, so for any m_α, $\alpha n_\alpha \notin \ell_R(m_\alpha) n_\alpha$. Also one easily checks that $\ell_R(M) \subseteq \ell_R(N)$ whenever M controls N.

The necessity part of the following corollary was established in [9], and has been employed in several subsequent papers. The second statement generalizes the fact that if M is reflexive and controls N, then $M \oplus N$ is reflexive.

Corollary 2.2. *The bimodule ${}_RM_\Delta$ controls ${}_RN_\Delta$ if and only if the restriction mapping*

$$res: \ alglat(M \oplus N) \to \ alglat(M)$$

is injective.

Moreover, if these conditions hold and $Im(res) \subseteq \lambda(R)$, then $M \oplus N$ is reflexive.

Proof. ($\Leftarrow$) Suppose M does not control N. Then there is an $\alpha \neq 0$ in $\mathrm{End}(N_\Delta)$ such that for all $m \in M$ and all $n \in N$, there is an $r \in \ell_R(m)$ with $\alpha n = rn$. But then $0 \neq \beta = (0, \alpha) \in \mathrm{End}((M \oplus N)_\Delta)$ and $\beta(m,n) = (0, \alpha n) = r(m,n)$, so $\beta \in \mathrm{alglat}(M \oplus N)$, and $\beta|_M = 0$.

($\Rightarrow$) If $\beta = (0, \alpha) \in \mathrm{alglat}(M \oplus N)$ with $\alpha \neq 0$, then for each pair $(m,n) \in M \times N$ there is an $r \in R$ with $\beta(m,n) = (rm, rn) = (0, rn)$, contrary to the condition of Proposition 2.1.

For the last statement, assume that res is injective, $\delta \in \mathrm{alglat}(M \oplus N)$ and $res(\delta) = \lambda_2(r)$ in the commutative diagram:

$$\begin{array}{ccc} \mathrm{alglat}(M \oplus N) & \stackrel{res}{\to} & \mathrm{alglat}(M) \\ {}_{\lambda_1} \nwarrow & & {}_{\lambda_2} \nearrow \\ & R & \end{array}$$

Then $res(\delta) = \lambda_2(r) = res(\lambda_1(r))$, so $\delta = \lambda_1(r)$. □

Suppose that M controls N. Then N is an $R/\ell_R(M)$-module, and if M is reflexive, then so is $M \oplus N$. Conversely we have

Corollary 2.3. *If ${}_RM$ is faithful and $M \oplus N$ is reflexive, then M controls N. Thus if M is a faithful reflexive R-module, then $M \oplus N$ is reflexive if and only if M controls N.*

Proof. As in the previous proof, we have $res \circ \lambda_1 = \lambda_2$. Thus if λ_1 is epic and λ_2 is monic, then res is monic, so Corollary 2.2 applies. □

A QF-3 algebra R is a finite dimensional algebra with a (*unique minimal faithful*) module U that embeds as a direct summand in every faithful module. In [9] it was shown that over a QF-2 algebra (a special type of QF-3 algebra, see [1, Section 31]) every faithful module is reflexive if U is reflexive; and the problem was posed of determining whether this is the case for QF-3 algebras. Some recent progress has been made by Snashall in [15]. Perhaps this last corollary may help to shed more light on this problem.

3. Universal k-reflexivity

In [2] Azoff showed, from an operator theory point of view, that for an integer $k \geq 3$, if $\dim({}_{\mathbb{C}}M)) \leq k$, then the direct sum $M^{(k-1)}$ of $k-1$ copies of M is reflexive. This topic was treated later using algebraic methods in [11]. We conclude by showing that for any left artinian ring R there is an integer k such that the direct sum of k copies of every bimodule ${}_RM_\Delta$ is reflexive. To do so we shall employ

Lemma 3.1. *If $R/\ell_R(M)$ embeds in N and ${}_RM_\Delta$ controls ${}_RN_\Delta$, then $M \oplus N$ is reflexive.*

Proof. Since $R/\ell_R(M)$ embeds in N, there is an $n_0 \in N$ with $\ell_R(n_0) = \ell_R(M)$. Let $\delta = (\beta, \gamma) \in \text{alglat}(M \oplus N)$ and suppose that $\gamma(n_0) = sn_0$. Then for any $m \in M$, there are $r, t \in R$ with

$$r(m, n_0) = \delta(m, n_0) = (\beta m, \gamma n_0) = (tm, sn_0),$$

so $r - s \in \ell_R(n_0) = \ell_R(M)$ and

$$\beta m = rm = sm.$$

Thus $res(\delta) = \beta = \lambda(s)$, and Corollary 2.2 applies. □

Theorem 3.2. *For each left artinian ring R there is a positive integer k such that every $R-\Delta$-bimodule is k-reflexive. Indeed if*

$$k = \sup\{c(Soc({}_RR/I)) \,|\, I \leq {}_RR_R\} + 1,$$

then $M^{(k)}$ is reflexive for every bimodule ${}_RM_\Delta$.

Proof. Let $c = c(Soc({}_RR/\ell_R(M)))$ and let $N = M^{(c)}$ Then it is easy to see that $R/\ell_R(M)$ embeds in N and, of course M controls N. Thus $M \oplus N = M^{(c+1)}$ is reflexive by Lemma 3.1. □

This value of k cannot be improved since, over the uniserial K-algebra $R = K[x]/x^2$, the regular module ${}_RR$ is not reflexive (see [2] or [9]).

According to [11, Corollary 3] the result of Azoff mentioned above can be extended to the K-algebra-bimodule case. Thus from this result and the proof of Theorem 3.2 we have

Corollary 3.3. *Let R and Δ be finite dimensional K-algebras and let M be an $R-\Delta$-bimodule such that ${}_RM$ is faithful. If $k \geq 2$ and*

$$min\{c(Soc({}_RR)) + 1,\, c(M_\Delta) - 1\} \leq k,$$

then $M^{(k)}$ is reflexive.

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