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A REMARK ON THE BERGMAN STABILITY

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ABSTRACT. Let $\{D_k\}, k = 1, 2, \cdots$, be a sequence of bounded pseudoconvex domains that converges, in the sense of Boas, to a bounded domain D. We show that if ∂D can be described locally as the graph of a continuous function in suitable coordinates for $\mathbf{C}^n$, then the Bergman kernel of D_k converges to the Bergman kernel of D uniformly on compact subsets of $D \times D$.

1. INTRODUCTION

Let D be a bounded domain in $\mathbf{C}^n$. By $K_D(z, w)$ we denote the Bergman kernel of D. After the early paper of Ramadanov [7], there is a long list of papers concerning the stability problem of the Bergman kernels of a sequence of domains $D_k \to D$ (cf. [1], [2], [3], [4], [5], [8]). The example [8] of decreasing concentric disks in the complex plane converging to a disk with a slit removed shows that it is not sufficient to require only that the D_k converge to D in the sense of Boas [1], i.e., the D_k eventually swallow every compact subset of D and are eventually swallowed by every open neighbourhood of $\overline{D}$. Boas [1] proved stability when D has C^2 boundary and D_k are pseudoconvex. He also asked if the hypothesis could be reduced to C^1 boundary regularity. The answer is yes; in fact, we are going to prove the following

Main Theorem. *Let $\{D_k\}$ be a sequence of bounded pseudoconvex domains that converges, in the sense of Boas, to a bounded domain D. Suppose that ∂D can be described locally as the graph of a continuous function in suitable coordinates for $\mathbf{C}^n$. Then the sequence $K_{D_k}(z, w)$ converges to $K_D(z, w)$ uniformly on compact subsets of $D \times D$.*

Proof of the Main Theorem. As in [1], it is sufficient to show that if f is a square-integrable holomorphic function on D, and if a positive ϵ is prescribed, then for all sufficiently large k there exists a square-integrable holomorphic function f_k on D_k such that $\|f_k - f\|_{L^2(D_k \cap D)} < \epsilon$ and $\|f_k\|_{L^2(D_k \setminus D)} < \epsilon$.

For each $\zeta^0 \in \partial D$, there is a neighbourhood U of ζ^0 such that $D \cap U = \{z \in U | x_{2n} < \psi(x_1, \cdots, x_{2n-1})$ in suitable coordinates $z = (x_1 + ix_2, \cdots, x_{2n-1} + ix_{2n}) \in \mathbf{C}^n$, where ψ is a continuous function. Then there exists a neighbourhood V of ζ^0 with $V \subset\subset U$ such that for all sufficiently small $\delta > 0$ one has $z - (0, \cdots, 0, i\delta) \in D$ for all $z \in \overline{D} \cap V$ and $z + (0, \cdots, 0, i\delta) \in \mathbf{C}^n \setminus \overline{D}$ for all $z \in (\mathbf{C}^n \setminus D) \cap V$. Hence

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we can assume that there exist finitely many points $\zeta_j \in \partial D$, $1 \le j \le l$, reals $\delta_0 > 0$, $r > 0$, and unit outward vectors N_j at ζ_j such that

$$(\overline{D} \cap B(\zeta_j, r)) - \delta N_j \subset D,$$
$$((\mathbf{C}^n \backslash D) \cap B(\zeta_j, r)) + \delta N_j \subset \mathbf{C}^n \backslash \overline{D}$$

for all $0 < \delta \le \delta_0$, where $B(\zeta, r)$ is the ball in $\mathbf{C}^n$ which is centered at ζ with radius r. Choose $U_0 \subset\subset D$ such that $U_0 \cup (B(\zeta_j, r))_{1 \le j \le l}$ cover $\overline{D}$. Let $(\varphi_j)_{0 \le j \le l}$ be a smooth partition of unity associated to the covering $U_0, B(\zeta_j, r)$. For each $0 < \delta \le \delta_0$ we put

$$h_\delta(z) = \sum_{j=1}^{l} f(z - \delta N_j)\varphi_j(z) + \varphi_0(z) f(z).$$

Then h_δ is C^∞ on an open neighbourhood of $\overline{D}$. For each $z \in D$, we have

$$h_\delta(z) = \sum_{j=1}^{l} (f(z - \delta N_j) - f(z))\varphi_j(z) + f(z),$$

which gives

$$\bar{\partial} h_\delta = \sum_{j=1}^{l} (f(z - \delta N_j) - f(z))\bar{\partial}\varphi_j.$$

Hence for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|\bar{\partial} h_\delta\|_{L^2(D)} \le \epsilon$ and $\|h_\delta - f\|_{L^2(D)} \le \epsilon$. Now fix δ. Then

$$\begin{aligned} \|\bar{\partial} h_\delta\|_{L^2(D_k)} &= \|\bar{\partial} h_\delta\|_{L^2(D_k \cap D)} + \|\bar{\partial} h_\delta\|_{L^2(D_k \backslash D)} \\ &\le \|\bar{\partial} h_\delta\|_{L^2(D)} + \|\bar{\partial} h_\delta\|_{L^2(D_k \backslash D)} \\ &\le \epsilon + \|\bar{\partial} h_\delta\|_{L^2(D_k \backslash D)} \\ &\le 2\epsilon \end{aligned}$$

for all sufficiently large k. By a well-known theorem of Hörmander (cf. [6]), there exists a function u_k on D_k such that $\bar{\partial} u_k = \bar{\partial} h_\delta$ and

$$\|u_k\|_{L^2(D_k)} \le C \|\bar{\partial} h_\delta\|_{L^2(D_k)} \le 2C\epsilon,$$

where C is a positive constant depending only on the diameter of D. Put $f_k = h_\delta - u_k$. Then f_k is a holomorphic function on D_k satisfying

$$\begin{aligned} \|f_k - f\|_{L^2(D_k \cap D)} &\le \|h_\delta - f\|_{L^2(D_k \cap D)} + \|u_k\|_{L^2(D_k)} \\ &\le (1 + 2C)\epsilon \end{aligned}$$

and

$$\begin{aligned} \|f_k\|_{L^2(D_k \backslash D)} &\le \|h_\delta\|_{L^2(D_k \backslash D)} + \|u_k\|_{L^2(D_k)} \\ &\le (1 + 2C)\epsilon \end{aligned}$$

for all sufficiently large k. Q.E.D.

In the case of $n = 1$, a sufficient and nessesary condition of Bergman stability given in [8] is the following.

Proposition. *Let* $\overline{D} = \bigcap_{k=1}^{\infty} D_k,\ D \subset D_{k+1} \subset D_k,\ D = \mathrm{int}\overline{D}$ *and* $|\partial D| = 0$. *Then* $K_{D_k}(z,w) \to K_D(z,w)$ *as* $k \to \infty$ *locally uniformly in* $(z,w) \in D \times D$ *if and only if the set of all* $z \in \partial D$ *which do not belong to the fine closure of the complement of* $\overline{D}$ *has a zero logarithmic capacity.*

Therefore, we can't expect a similar phenomenon in the theorem to hold on a bounded domain which is only assumed to be the interior of its closure.

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