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REIDEMEISTER TORSION OF T^2-BUNDLES OVER S^1 FOR $SL(2;\mathbf{C})$-REPRESENTATIONS

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ABSTRACT. We compute the $SL(2,\mathbf{C})$-Reidemeister torsion of torus bundles over S^1 which monodromies are hyperbolic elements in $SL(2,\mathbf{Z})$.

0. INTRODUCTION

In [1], Johnson studied the Reidemeister torsion of 3-manifolds for $SL(2,\mathbf{C})$ irreducible representations. In our previous paper [2], following this line, we obtained an explicit formula of the Reidemeister torsion of Seifert fibered spaces for $SL(2;\mathbf{C})$ irreducible representations. In particular the sets of values of the Reidemeister torsion are finite subsets in $\mathbf{R}$. It follows that it has no continuous variations, even if the dimension of the space of the representations of the fundamental group is positive. On the other hand, in the case of the figure-eight knot exterior, we showed in [3] that it has continuous variations. We would like to understand the Reidemeister torsion of 3-manifolds for $SL(2;\mathbf{C})$-representations from a viewpoint of geometric structures. It is well known that a Seifert fibered space admits only one of 6 geometric structures, and the complement of the figure-eight knot admits a complete hyperbolic structure.

In this paper, we study the T^2-bundles over S^1 with hyperbolic holonomies. It is well known that they are solvable manifolds. As a result, there exists only finite $SL(2,\mathbf{C})$ irreducible representations, and the value of Reidemeister torsion is a finite set. Combining with previous results, it seems that the Reidemeister torsion for $SL(2,\mathbf{C})$-representations is related with hyperbolic structures of 3-manifolds.

Now we describe the contents of this paper briefly. In section 1, we give a definition of the Reidemeister torsion. In section 2, we compute irreducible representations of T^2-bundles. In section 3, we prove the formula about Reidemeister torsion.

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1. Definition of Reidemeister torsion

First let us describe the definition of the Reidemeister torsion for $SL(2;\mathbf{C})$-representations. See Johnson [1] and Kitano [2], [3] for details.

Let S be an n-dimensional vector space over $\mathbf{C}$ and let $\mathbf{b}=(b_1,\dots,b_n)$ and $\mathbf{c}=(c_1,\dots,c_n)$ be two bases for S. Setting $b_i = \sum_{j=1}^{n} p_{ij}c_j$, we obtain a nonsingular matrix $P=(p_{ij}) \in GL(n,\mathbf{C})$. Let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of P.

Suppose

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is an acyclic chain complex of finite-dimensional vector spaces over $\mathbf{C}$. We assume that a preferred basis $\mathbf{c}_q$ for $C_q(C_*)$ is given for each q. Choose some basis $\mathbf{b}_q$ for $B_q(C_*)$ and take a lift of it in $C_{q+1}(C_*)$, which we denote by $\tilde{\mathbf{b}}_q$.

Since $B_q(C_*) = Z_q(C_*)$, the basis $\mathbf{b}_q$ can serve as a basis for $Z_q(C_*)$. Furthermore the sequence

$$0 \to Z_q(C_*) \to C_q(C_*) \to B_{q-1}(C_*) \to 0$$

is exact and the vectors $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ form a basis for $C_q(C_*)$.

Definition 1.1. The torsion $\tau(C_*)$ of the chain complex C_* is given by the alternating product

$$\prod_{q=0}^{m} [\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]^{(-1)^{q+1}}.$$

Remark. It is easily shown that this torsion does not depend on the choice of the lift $\tilde{\mathbf{b}}_{q-1}$. Furthermore it does not depend on the choice of $\{\mathbf{b}_q\}$. See Milnor [7].

Let X be a finite cell complex and let $\tilde{X}$ be a universal covering of X. The fundamental group $\pi_1 X$ acts on $\tilde{X}$ as deck transformations. Then the chain complex $C_*(\tilde{X};\mathbf{Z})$ has the structure of a chain complex of free $\mathbf{Z}[\pi_1 X]$-modules. Let $\rho : \pi_1 X \to SL(2;\mathbf{C})$ be a representation. We simply write V to $\mathbf{C}^2$. Using the representation ρ, V has a structure of a $\mathbf{Z}[\pi_1 X]$-module and then we write V_ρ to it. Define the chain complex $C_*(X;V_\rho)$ by $C_*(\tilde{X};\mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis

$$\{\sigma_1 \otimes \mathbf{e}_1, \sigma_1 \otimes \mathbf{e}_2, \dots, \sigma_{k_q} \otimes \mathbf{e}_1, \sigma_{k_q} \otimes \mathbf{e}_2\}$$

of $C_q(X;V_\rho)$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a canonical basis of V and $\sigma_1,\dots,\sigma_{k_q}$ are q-cells giving the preferred basis of $C_q(\tilde{X};\mathbf{Z})$.

When $C_*(X;V_\rho)$ is acyclic (namely all homology groups vanish; $H_*(X;V_\rho)=0$), then we call ρ an acyclic representation.

Definition 1.2. Let $\rho : \pi_1 X \to SL(2;\mathbf{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(X)$ of X with V_ρ-coefficients is defined to be the torsion $\tau(C_*(X;V_\rho))$ of the chain complex.

Remark. (1) The Reidemeister torsion $\tau_\rho(X)$ seems to depend on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant. See Johnson [1] and Milnor [5], [7].

(2) If ρ is not acyclic, we define $\tau_\rho(X)$ to be zero.

The next propositions are well known but important. See Johnson [1], Kitano [2] for examples.

Proposition 1.3. *Let M be a closed, oriented 3-manifold with torus decomposition $A \cup_{T^2} B$ and let $\rho : \pi_1 M \to SL(2; \mathbf{C})$ be a representation whose restriction to $\pi_1 T^2$ is acyclic. Then $H_*(M; V_\rho) = 0$ if and only if $H_*(A; V_\rho) = H_*(B; V_\rho) = 0$. Moreover in this case*

$$\tau_\rho(M) = \tau_\rho(A)\tau_\rho(B).$$

Proposition 1.4. *Let $\rho : \pi_1 S^1 \to SL(2; \mathbf{C})$ be a representation. Let h be a generator of $\pi_1(S^1)$. Then*

$$\tau_\rho(S^1) = 1/det(\rho(h) - I).$$

2. $SL(2, \mathbf{C})$-representation of a T^2-bundle

Let M_A be a T^2-bundle over S^1 whose holonomy matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2; \mathbf{Z})$. We assume $|tr(A)| = |a + d| \geq 3$. We call such a matrix a hyperbolic element in $SL(2; \mathbf{Z})$. The presentation of the fundamental group $\pi_1(M_A)$ is given by the following:

$$\pi_1(M_A) = \langle x, y, h \,|\, x^a y^c = hxh^{-1}, x^b y^d = hyh^{-1}, xyx^{-1}y^{-1} = 1 \rangle.$$

Notation. When a representation $\rho : \pi_1(M_A) \to SL(2; \mathbf{C})$ is fixed, we denote the matrix $\rho(x)$ for $x \in \pi_1(M_A)$ by the corresponding capital letter X.

Let $\rho : \pi_1(M_A) \to SL(2; \mathbf{C})$ be an irreducible representation. Since the subgroup $\pi_1(T^2) = \langle x, y |\, xyx^{-1}y^{-1} \rangle$ is abelian, we can assume that the pair of X and Y is the one of the following 3 cases, up to conjugate. Here $s, t \in \mathbf{C} - \{0\}$ and $u \in \mathbf{C}$.

(1) $X = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, Y = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

(2) $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$.

(3) $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} -1 & u \\ 0 & -1 \end{pmatrix}$.

In cases (2) and (3), we can see that H is an upper triangle matrix whose diagonal entries are -1 or 1 by computing the relations $x^a y^c = hxh^{-1}$ and $x^b y^d = hyh^{-1}$. Then this representation ρ is reducible. Then we consider only case (1). Similar to the above cases, from the relations $x^a y^c = hxh^{-1}$ and $x^b y^d = hyh^{-1}$, H is not a diagonal matrix because ρ is irreducible. Then we may assume that $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ up to conjugate. Hence as the relations of s and t, we have $s^{a+1} t^c = 1$ and $s^b t^{d+1} = 1$ and may assume that

$$X = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, Y = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to see that $\{(s, t) |\, s^{a+1} t^c = 1,\ s^b t^{d+1} = 1\} = \{(s, t) |\, s^{a+d+1} = 1,\ s^d = t^c\}$ is a finite set.

3. Formula of Reidemeister torsion

We fix a 2-disk D on T^2. As a mapping class on T^2, we can assume A is the identity on D. Then M_A can be decomposed to $N = D \times S^1$ and $\hat{M}_A = M_A \cup intN$. Now we state the main theorem.

Theorem 3.1.

$$\begin{aligned}\tau_\rho(M_A) =& \frac{1}{4}\{1 + f(s,a)f(s^{-1},a) + f(t,d)f(t^{-1},d) \\ &+ f(s,a)f(s^{-1},a)f(t,d)f(t^{-1},d) + f(s,b)f(s^{-1},b)f(t,c)f(t^{-1},c) \\ &+ s^{-a}f(s,b)f(t^{-1},c) + s^a f(s^{-1},b)f(t,c) \\ &- s^{a-b}f(s^{-1},a)f(s,b)f(t,c)f(t^{-1},d) \\ &- s^{b-a}f(s,a)f(s^{-1},b)f(t^{-1},c)f(t,d)\},\end{aligned}$$

where

$$f(z,\alpha) = \begin{cases} \sum\limits_{i=0}^{\alpha-1} z^i, & \alpha > 0, \\ -\sum\limits_{i=1}^{|\alpha|} z^{-i}, & \alpha < 0, \\ 1, & \alpha = 0. \end{cases}$$

Remark. This formulae holds without the assumption that A is a hyperbolic element.

Proof. Applying Proposition 1.4, we have $\tau_\rho(N) = \det(H - I) = \frac{1}{2}$ because N collapses to S^1. Now it is easy to see that the restriction of ρ to $\partial N \cong T^2$ is acyclic. See Kitano [2] for the proof. Hence by Proposition 1.3 and the above, we obtain

$$\tau_\rho(M_A) = \frac{1}{2}\tau_\rho(\hat{M}_A).$$

To compute $\tau_\rho(\hat{M}_A)$, we take a 2-dimensional complex W which is constructed from three 1-cells x, y, h and two 2-cells D_1, D_2 with attaching maps given by $x^a y^c h x^{-1} h^{-1}$ and $x^b y^d h y^{-1} h^{-1}$. Since it is easy to see that $\hat{M}_A$ collapses to W, then we compute the torsion of the chain complex of W,

$$C_*(W; V_\rho) : 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where

$$\begin{aligned}\partial_2 &= \begin{pmatrix} f(X,a) - X^aY^cHX^{-1} & f(X,b) \\ X^a f(Y,c) & X^b f(Y,d) - X^bY^dHY^{-1} \\ X^aY^c - X^aY^cHX^{-1}H^{-1} & X^bY^d - X^bY^dHY^{-1}H^{-1} \end{pmatrix} \\ &= \begin{pmatrix} f(s,a) & -1 & f(s,b) & 0 \\ 1 & f(s^{-1},a) & 0 & f(s^{-1},b) \\ f(t,c)s^a & 0 & f(t,d)s^b & -1 \\ 0 & f(t^{-1},c)s^{-a} & 1 & f(s^{-1},d)s^{-b} \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \\ \partial_1 &= \begin{pmatrix} X - I & Y - I & H - I \end{pmatrix}.\end{aligned}$$

Now we write $f(X,a)$ to the matrix inputting X to $f(z,a)$. For simplicity we put

$$T = \begin{pmatrix} f(s,a) & -1 & f(s,b) & 0 \\ 1 & f(s^{-1},a) & 0 & f(s^{-1},b) \\ f(t,c)s^a & 0 & f(t,d)s^b & -1 \\ 0 & f(t^{-1},c)s^{-a} & 1 & f(t^{-1},d)s^{-b} \end{pmatrix}.$$

By the definition of $\tau_\rho(\hat{M}_A)$, it is easy to see that $\tau_\rho(\hat{M}_A) = det(T)/det(H-I)$. Therefore we obtain

$$\begin{aligned} \tau_\rho(\hat{M}_A) = &\frac{1}{2}det(T) \\ =&\frac{1}{2}\{1 + f(s,a)f(s^{-1},a) + f(t,d)f(t^{-1},d) \\ &+ f(s,a)f(s^{-1},a)f(t,d)f(t^{-1},d) + f(s,b)f(s^{-1},b)f(t,c)f(t^{-1},c) \\ &\quad + s^{-a}f(s,b)f(t^{-1},c) + s^a f(s^{-1},b)f(t,c) \\ &- s^{a-b}f(s^{-1},a)f(s,b)f(t^{-1},d)f(t,c) \\ &- s^{b-a}f(s^{-1},b)f(s,a)f(t^{-1},c)f(t,d)\}. \end{aligned}$$

Therefore we obtain the formula of the Reidemeister torsion. □

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