

THE MARDEŠIĆ FACTORIZATION THEOREM FOR EXTENSION THEORY AND C-SEPARATION

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ABSTRACT. We shall prove a type of Mardešić factorization theorem for extension theory over an arbitrary stratum of CW-complexes in the class of arbitrary compact Hausdorff spaces. Our result provides that the space through which the factorization occurs will have the same strong countability property (e.g., strong countable dimension) as the original one had. Taking into consideration the class of compact Hausdorff spaces, this result extends all previous ones of its type. Our factorization theorem will simultaneously include factorization for weak infinite-dimensionality and for Property C, that is, for C-spaces.

A corollary to our result will be that for any weight α and any finitely homotopy dominated CW-complex K , there exists a Hausdorff compactum X with weight $wX \leq \alpha$ and which is universal for the property $X\tau K$ and weight $\leq \alpha$. The condition $X\tau K$ means that for every closed subset A of X and every map $f : A \rightarrow K$, there exists a map $F : X \rightarrow K$ which is an extension of f . The universality means that for every compact Hausdorff space Y whose weight is $\leq \alpha$ and for which $Y\tau K$ is true, there is an embedding of Y into X .

We shall show, on the other hand, that there exists a CW-complex S which is not finitely homotopy dominated but which has the property that for each weight α , there exists a Hausdorff compactum which is universal for the property $X\tau S$ and weight $\leq \alpha$.

1. INTRODUCTION

In this paper we shall prove the following result.

1.1. Theorem (Factorization Theorem for Extension Theory and C-Separation). *Let $f : X \rightarrow Y$ be a map of compact Hausdorff spaces X and Y , and suppose that \mathcal{A} is a countable collection of closed subsets of X . Then there exists a compact Hausdorff space Z , a surjective map $g : X \rightarrow Z$, and a map $p : Z \rightarrow Y$ such that*

- (a) $wZ \leq \max\{wY, \aleph_0\}$,
- (b) $p \circ g = f$,
- (c) *if N is a CW-complex, $D \in \mathcal{A}$, A is a closed subset of $g(D)$, $f_0 : A \rightarrow N$ is a map, and there exists a map $F : D \rightarrow N$ such that $F(x) = f_0(g(x))$ for all*

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- $x \in g^{-1}(A) \cap D$, then there is a map $H : g(D) \rightarrow N$ which is an extension of f_0 , and
- (d) if \mathcal{S} is a sequence of open covers of Z , $D \in \mathcal{A}$, and $g^{-1}(\mathcal{S})|D$ has a C -refinement in D , then $\mathcal{S}|g(D)$ has a C -refinement in $g(D)$.

We shall provide a proof in section 2.

1.2. Definition. Whenever X, Y, Z are compact Hausdorff spaces, $f : X \rightarrow Y$, $g : X \rightarrow Z$, $p : Z \rightarrow Y$ are maps, g is surjective, and $p \circ g = f$, we shall refer to (g, p) as a **factorization** of f through Z .

By definition, for a space X and a CW-complex K , $X\tau K$ (sometimes written $\dim X \leq K$) means that for each closed subset A of X and map $f : A \rightarrow K$, there exists a map $F : X \rightarrow K$ which is an extension of f . (By map, we always mean continuous function). This is the basic notion of extension theory [DD]. Each dimension theory, covering dimension \dim or cohomological dimension \dim_G over an abelian group G , can be expressed in terms of extension theory. For it is well known ([Na]) that for paracompact spaces X , $\dim X \leq n$ if and only if $X\tau S^n$ and that $\dim_G X \leq n$ (for an arbitrary abelian group G) if and only if $X\tau K$ for each Eilenberg-MacLane complex K of type $K(G, n)$.

Theorem 1.1 leads to the following corollary which captures the original Mardešić Factorization Theorem ([Ma1]) as well as all of its direct successors for the class of compact Hausdorff spaces.

1.3. Corollary. *Let $f : X \rightarrow Y$ be a map of compact Hausdorff spaces, (g, p) be a factorization of f through a compact Hausdorff space Z with $wZ \leq \max\{wY, \aleph_0\}$ as in Theorem 1.1, and assume that $X \in \mathcal{A}$. If K is an arbitrary CW-complex and $X\tau K$, then $Z\tau K$. Consequently, if $\dim X \leq n$, then $\dim Z \leq n$; if G is an abelian group and $\dim_G X \leq n$, then $\dim_G Z \leq n$. \square*

The first version of the Mardešić Factorization Theorem appeared in [Ma1] where it was designed for dimension theory \dim (see also [Ma2]). The work in [Pa] (Corollary 2) improves that in [Ma1], yielding a theorem like our Corollary 1.3 but with the restriction that $wK \leq wY$ (see also [Dr], page 472, Factorization Theorem). Proposition 4.9 of [Ch] contains a factorization theorem for Tychonoff spaces which uses as “complexes”, ANR Polish spaces. Since every countable CW-complex is homotopy equivalent to a Polish ANR, then that result when restricted to the class of compact Hausdorff spaces contains the result of [Dr], at least for countable complexes. We refer the reader to section 2 of [Dy1] for several results concerning factorization in the class of separable metrizable spaces. Other authors such as A. Arhangel’skiĭ, W. Luxemburg and A. Zarelua have made significant contributions to this subject.

A nice aspect of our factorization is that it can preserve a property such as strong countable dimension. Let us state a corollary which makes this explicit.

1.4. Corollary. *Let $f : X \rightarrow Y$ be a map of compact Hausdorff spaces, and assume that X_i are closed subspaces of X for each $i \in \mathbb{N}$. Then there is a factorization (g, p) of f through a compact Hausdorff space Z with $wZ \leq \max\{wY, \aleph_0\}$ such that if K is a CW-complex and $X_i\tau K$, then $g(X_i)\tau K$. In the notation of [DD], this could be stated as*

$$\text{ext-dim}(g(X_i)) \leq \text{ext-dim}(X_i). \quad \square$$

The notions, weak infinite-dimensionality and C-space (called Property C in some references) are defined in [En2]. For compact Hausdorff spaces, as required herein, there is a more concise, unified approach to these concepts.

1.5. Definition. Let $\mathcal{S} = \{\mathcal{S}_i \mid i \in \mathbb{N}\}$ be a sequence of families of subsets of a space X . Then we say that \mathcal{S} is finite if for almost all i , $\mathcal{S}_i = \emptyset$. A sequence $\mathcal{C} = \{\mathcal{C}_i \mid i \in \mathbb{N}\}$ of families of subsets of X refines \mathcal{S} if \mathcal{C}_i refines \mathcal{S}_i for each i , and is called a C-refinement if in addition each \mathcal{C}_i is closed and discrete and $\bigcup \mathcal{C}$ covers X .

The reader should be able to construct a proof of the next statement.

1.6. Lemma. *A compact Hausdorff space X is a C-space (weakly infinite-dimensional space) if every sequence of open (two-element open) covers of X admits a C-refinement.* \square

This yields a third corollary to Theorem 1.1.

1.7. Corollary. *Let $f : X \rightarrow Y$ and Z be as in Theorem 1.1 and assume that $X \in \mathcal{A}$. If X is a C-space, then so is Z . If X is weakly infinite-dimensional, then so is Z .* \square

1.8. Definition. Let K be a CW-complex, \mathcal{C} a class of spaces, and $X \in \mathcal{C}$. Then we say that X is **universal** for K -extension theory, if $X\tau K$, and for each $Y \in \mathcal{C}$ with $Y\tau K$, there is an embedding of Y into X .

The next result can be obtained from previous work such as [Dr] (see also [Dy2], [DM] for related material), but it also follows readily from Theorem 1.1, so we will indicate a proof here. The existence (or lack of existence) of universal compacta even in case K is of type $K(\mathbb{Z}, n)$ and $\alpha = \aleph_0$ is not known.

1.9. Corollary. *For every finitely homotopy dominated CW-complex K and every infinite cardinal α , there exists a compact Hausdorff space X which is universal for the class of Tychonoff spaces of weight $\leq \alpha$ and K -extension theory.*

Proof. Let us use βY to designate the Stone-Ćech compactification of a Tychonoff space Y . It is easy to check that if K is a CW-complex and the space Y is the topological sum of a collection $\{Y_a \mid a \in A\}$ of Tychonoff spaces with $Y_a\tau K$ for each $a \in A$, then Y is a Tychonoff space and $Y\tau K$. The result now follows from application of Lemma 1.10 below, 7.4.14 and 7.4.16 of [En1] and Theorem 1.1. \square

1.10. Lemma. *Let K be a finitely homotopy dominated CW-complex and let Y be a Tychonoff space such that $Y\tau K$. Then $\beta Y\tau K$.*

Proof. Let $A \subset \beta Y$ be closed and $f : A \rightarrow K$ be a map. Applying Lemma 2.11 of [RS], one may find a closed neighborhood N of A in βY such that $\text{Cl}_{\beta Y}(N \cap Y) = N$ and f extends to a map $f_0 : N \rightarrow K$. Since $Y\tau K$, there exists a map $F : N \cup Y \rightarrow K$ such that $F|N = f_0 : N \rightarrow K$. Since K is finitely dominated, F is homotopic to a map $G : N \cup Y \rightarrow K$ where $G(N \cup Y) \subset K_0 \subset K$ and K_0 is compact. Let $H : \beta Y \rightarrow K_0$ be the unique continuous extension of $G|Y : Y \rightarrow K_0$. Then of course $G|A = H|A : A \rightarrow K_0 \subset K$. Since $f = F|A \simeq G|A = H|A$ and $H|A$ extends to $H : \beta Y \rightarrow K$, then the homotopy extension property shows us that f extends to a map of βY to K . \square

To help round out the discussion, let us give a simple example.

1.11. Lemma. *There exists a CW-complex S which is not finitely homotopy dominated, but which has the property that for each weight α , there is a Hausdorff compactum X which is universal for S -extension theory, and compact Hausdorff spaces of weight $\leq \alpha$.*

Proof. Define $S = [1/2, \infty) \cup \bigcup_{i=1}^{\infty} S_i \subset \mathbb{R}^2$ where S_i is the circle centered at $(i, 1/2) \in \mathbb{R}^2$ whose radius is $1/2$. Then S has the structure of a connected CW-complex and $\pi_1(S)$ is free on a countably infinite set of generators. One can see from this fact that S is not finitely homotopy dominated.

Since S^1 is homeomorphic to a retract of S , then it is easy to detect that $X\tau S$ implies $X\tau S^1$. Conversely, suppose X is a compact Hausdorff space and $X\tau S^1$. Then $\dim X \leq 1$. From this perspective it is not difficult to see that any map of a closed subset of X to a connected CW-complex K extends to a map of X to K . Hence $X\tau S$.

We have demonstrated that for any compact Hausdorff space X , $X\tau S^1$ if and only if $X\tau S$. By Corollary 1.9, for any α there exists a Hausdorff compactum which is universal for S -extension theory and weight α . (Of course, $X\tau S^1$ simply means that $\dim X \leq 1$, and the existence of universal compacta for \dim has long been known [En1].) \square

Every CW-complex K_0 is homotopy equivalent to $|K|$ for some simplicial complex K (all polyhedra $|K|$ are given the weak topology herein). Since it is convenient for us to work in the class of simplicial complexes and their polyhedra, all our proofs in section 2 will be in terms of these objects instead of CW-complexes.

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2. PROOF OF THE FACTORIZATION THEOREM

We first make a definition which will be convenient for our work below.

2.1. Definition. Let X, Y, Z be spaces, D a subset of X , and suppose we are given maps $f : X \rightarrow Y$, $g : X \rightarrow Z$, and $p : Z \rightarrow Y$ such that $p \circ g = f$. Then we shall say that the triple (f, g, p) is **lifting for simplicial complexes relative to D** if for each closed subset A of Y , each simplicial complex N and each pair of maps $h : A \rightarrow |N|$, $F : D \rightarrow |N|$ having the property that $F(x) = h(f(x))$ for all $x \in D \cap f^{-1}(A)$, there exists a map $H : g(D) \rightarrow |N|$ such that

$$H(z) = h(p(z)) \text{ for all } z \in g(D) \cap p^{-1}(A).$$

We call (f, g, p) **C-lifting relative to D** if for each sequence \mathcal{S} of open covers of Y such that the sequence $f^{-1}(\mathcal{S})|D$ has a C-refinement in D , the sequence $p^{-1}(\mathcal{S})|g(D)$ has a C-refinement in $g(D)$.

Our proof of Theorem 1.1 is based on the next lemma whose proof will be provided later.

2.2. Lemma. *Let X, Z_1 be compact Hausdorff spaces, $D \subset X$ be closed, and $f_1 : X \rightarrow Z_1$ be a map. Then there exists a compact Hausdorff space Z_2 with $wZ_2 \leq \max\{wZ_1, \aleph_0\}$, a surjective map $f_2 : X \rightarrow Z_2$, and a map $p : Z_2 \rightarrow Z_1$ such that $p \circ f_2 = f_1$ and (f_1, f_2, p) is simultaneously lifting for simplicial complexes relative to D and C-lifting relative to D .*

We shall also use the next two facts; but we leave their proofs to the reader.

2.3. Lemma. *Let $g : X \rightarrow Z$ be a map of compact Hausdorff spaces, N be a finite simplicial complex, A be a closed subspace of Z , and $f_0 : A \rightarrow |N|$, $F : X \rightarrow |N|$ be maps such that $F(x) = f_0(g(x))$ for all $x \in g^{-1}(A)$. Then there exists a closed neighborhood B of A in Z and a map $\tilde{f}_0 : B \rightarrow |N|$ extending f_0 , such that, with $B_0 = g^{-1}(B)$, $F|_{B_0} : B_0 \rightarrow |N|$ is homotopic to $\tilde{f}_0 \circ g|_{B_0} : B_0 \rightarrow |N|$. \square*

2.4. Lemma. *Let X be a compact Hausdorff space and let \mathcal{S} be a sequence of open covers of X which has a C-refinement in X . Then \mathcal{S} has a C-refinement in X which is finite. \square*

Proof of Theorem 1.1. Let $Z_1 = f(X)$, $f_1 = f : X \rightarrow Z_1$, and write $\mathcal{A} = \{D_1, D_2, \dots\}$. Let $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ be a surjection each of whose point inverses is infinite. Apply Lemma 2.2 recursively to obtain a sequence $(Z_i)_{i=1}^\infty$ of compact Hausdorff spaces, two sequences $(f_i : X \rightarrow Z_i)_{i=1}^\infty$, $(p_{i+1} : Z_{i+1} \rightarrow Z_i)_{i=1}^\infty$ of maps, the f_i 's being surjective, such that for each i ,

- (1) $wZ_{i+1} \leq \max\{wZ_i, \aleph_0\}$,
- (2) $p_{i+1} \circ f_{i+1} = f_i$, and
- (3) (f_i, f_{i+1}, p_{i+1}) is lifting for simplicial complexes relative to $D_{\gamma(i)}$ and simultaneously is C-lifting relative to $D_{\gamma(i)}$.

Put $Z = \lim(Z_i, p_{i+1}, \mathbb{N})$. Then Z is a compact Hausdorff space and (1) shows that $wZ \leq \max\{wY, \aleph_0\}$. Using (2), let $g : X \rightarrow Z$ be the limit of the compatible system of surjective maps $f_i : X \rightarrow Z_i$. Then g is easily seen to be surjective. Let $p_i : Z \rightarrow Z_i$ denote coordinate projection, and define $p = p_1 : Z \rightarrow Z_1 \subset Y$. Then of course, $p \circ g = f$. Fix $D = D_k \in \mathcal{A}$ and let $D_0 = g(D)$, and $g_0 = g|_D : D \rightarrow D_0$.

We want to show that (c) of 1.1 is true. Let A be a closed subset of D_0 , N_0 be a CW-complex, and $f_0 : A \rightarrow N_0$, $F : D \rightarrow N_0$ be maps such that $F(x) = f_0(g(x)) = f_0(g_0(x))$ for all $x \in g^{-1}(A) \cap D = g_0^{-1}(A)$. The reader will easily see that there is no loss of generality in replacing N_0 by $|N|$, where N is a finite simplicial complex. We have to prove the existence of a map of D_0 to $|N|$ which is an extension of f_0 .

Using Lemma 2.3, extend f_0 to a map $\tilde{f}_0 : B \rightarrow |N|$, where B is a closed neighborhood of A in D_0 in such a manner that $F|_{B_0} : B_0 \rightarrow |N|$ is homotopic to $\tilde{f}_0 \circ g_0|_{B_0} : B_0 \rightarrow |N|$, where $B_0 = g_0^{-1}(B)$.

One may choose j so large that for each $i \geq j$, $p_i^{-1}(p_i(A)) \cap D_0 \subset B$. Hence there exists i such that $k = \gamma(i)$, and a map $h : p_i(B) \rightarrow |N|$ such that $h \circ p_i|_B \simeq \tilde{f}_0$ and $\tilde{A} = p_i^{-1}(p_i(A)) \cap D_0 \subset B$. Note that $A \subset \tilde{A}$. Since $p_i \circ g = f_i$, then $A' = D \cap f_i^{-1}(p_i(A)) = g_0^{-1}(\tilde{A}) \subset g_0^{-1}(B) = B_0$.

We wish to show that $\tilde{f}_0 \circ g_0|_{A'} : A' \rightarrow |N|$ is homotopic to $h \circ f_i|_{A'} : A' \rightarrow |N|$. Note that since $h \circ p_i|_B \simeq \tilde{f}_0$, then $h \circ p_i|_B \circ g_0|_{A'}$ is homotopic to $\tilde{f}_0 \circ g_0|_{A'}$ as maps of A' to $|N|$. But $h \circ p_i|_B \circ g_0|_{A'} = h \circ f_i|_{A'}$, which gives us the desired homotopy.

Since $\tilde{f}_0 \circ g_0|_{A'}$ is homotopic to $F|_{A'}$ and of course F is a map of D to $|N|$, then $h \circ f_i|_{A'}$ extends to a map $F' : D \rightarrow |N|$.

Consider the closed subset $p_i(A)$ of Z_i . Then for any $x \in A' = D \cap f_i^{-1}(p_i(A))$, $F'(x) = h(f_i(x))$. So we apply (3) to get a map $H : f_{i+1}(D) \rightarrow |N|$ such that $H(z) = h(p_{i+1}(z))$ for all $z \in f_{i+1}(D) \cap p_{i+1}^{-1}(p_i(A))$. If $a \in A$, then $p_{i+1}(a) = p_{i+1}(g(d)) = f_{i+1}(d)$ for some $d \in D$. Our proof will be complete if we can show that $H \circ p_{i+1}|_A : A \rightarrow |N|$ is homotopic to $f_0 : A \rightarrow |N|$. Clearly, $p_{i+1}(a) \in p_{i+1}^{-1}(p_i(A))$

since $p_{i+1} \circ p_i = p_i$. Therefore, $H \circ p_{i+1}(a) = h \circ p_{i+1} \circ p_i(a) = h \circ p_i(a)$, and $h \circ p_i|A \simeq \tilde{f}_0|A = f_0$.

Having obtained (c) of 1.1, it remains only to prove part (d). Let \mathcal{S} be a sequence of open covers of Z such that $g^{-1}(\mathcal{S})|D$ has a C-refinement say \mathcal{C} in D . By Lemma 2.4 we may as well assume that \mathcal{C} is finite.

It is then not difficult to show that there exists j so large that for each $i \geq j$ there is a sequence \mathcal{T} of open covers of Z_i such that $p_i^{-1}(\mathcal{T})$ refines \mathcal{S} and \mathcal{C} refines $f_i^{-1}(\mathcal{T})|D$. Choose $i \geq j$ so that $k = \gamma(i)$ and let \mathcal{T} be a sequence of open covers of Z_i as just indicated. By the construction, the sequence $p_{i+1}^{-1}(\mathcal{T})|f_{i+1}(D)$ has a C-refinement \mathcal{D} in $f_{i+1}(D)$. Thus $p_{i+1}^{-1}(\mathcal{D})|g(D)$ is a C-refinement of $\mathcal{S}|D$, so our proof is complete. □

Now let us prepare for a proof of Lemma 2.2.

2.5. Definition. Let Y be a compact Hausdorff space. A triple (A, h, N) will be called **allowable** for Y if A is a closed subspace of Y , N is a finite simplicial complex, and $h : A \rightarrow |N|$ is a map. An indexed family $\mathcal{F} = \{(A_\alpha, h_\alpha, N_\alpha)\}_{\alpha \in \Sigma}$ of allowable triples is said to be **dense** for Y if for each closed subspace A of Y , closed neighborhood G of A in Y , finite simplicial complex N , and map $h : G \rightarrow |N|$, there is an $\alpha \in \Sigma$ such that $A \subset A_\alpha \subset G$, and a simplicial isomorphism $\Phi_\alpha : N \rightarrow N_\alpha$ having the property that $\Phi_\alpha \circ h|A_\alpha \simeq h_\alpha$ as maps of A_α to $|N_\alpha|$.

We give the next lemma without proof.

2.6. Lemma. *Each compact Hausdorff space Y has a dense family $\mathcal{F} = \{(A_\alpha, h_\alpha, N_\alpha)\}_{\alpha \in \Sigma}$ such that $\text{card } \Sigma \leq \max\{wY, \aleph_0\}$.* □

Proof of Lemma 2.2. Applying the above Lemma 2.6, fix a dense family $\mathcal{F} = \{(A_\alpha, h_\alpha, N_\alpha)\}_{\alpha \in \Sigma}$ for Z_1 , $|N_\alpha| \subset Q_\alpha = Q$, the Hilbert cube, having the property that $\text{card } \Sigma \leq \max\{wZ_1, \aleph_0\}$, and for each $\alpha \in \Sigma$, let $D_\alpha = D \cap f_1^{-1}(A_\alpha)$. Denote by Σ' the set of indexes $\alpha \in \Sigma$ such that the map

$$h_\alpha \circ f_1|D_\alpha : D_\alpha \rightarrow |N_\alpha|$$

extends to a map $g_\alpha : X \rightarrow Q_\alpha$ with $g_\alpha(D) \subset |N_\alpha|$.

Fix also a family \mathcal{N} of open subsets of Z_1 such that $\text{card } \mathcal{N} \leq \max\{wZ_1, \aleph_0\}$ and for every closed subset A of Z_1 and neighborhood G of A , there exists an element U of \mathcal{N} such that $A \subset U \subset G$. Denote by $\mathcal{S}_\mathcal{N}$ the family of finite sequences of finite covers of Z_1 by elements of \mathcal{N} . Clearly $\text{card } \mathcal{S}_\mathcal{N} \leq \max\{wZ_1, \aleph_0\}$.

Let $\mathcal{S}'_\mathcal{N}$ consist of all $\mathcal{S} \in \mathcal{S}_\mathcal{N}$ such that $f_1^{-1}(\mathcal{S})|D$ has a C-refinement in D . For $\mathcal{S} \in \mathcal{S}'_\mathcal{N}$, let $\mathcal{C}_\mathcal{S}$ be a finite C-refinement of $f_1^{-1}(\mathcal{S})|D$ in D such that the (finite) families of $\mathcal{C}_\mathcal{S}$ consist of G_δ -sets.

For $\mathcal{S} \in \mathcal{S}'_\mathcal{N}$, the family $\mathcal{M} = \bigcup \mathcal{C}_\mathcal{S}$ is finite. For each nonempty $A \in \mathcal{M}$, take a map $f_A : X \rightarrow [0, 1]$ such that $f_A^{-1}(f_A(A)) = A$ and let $f_\mathcal{S} = \prod \{f_A | A \in \mathcal{M}\} : X \rightarrow [0, 1]^{\text{card } \mathcal{M}} \subset Q$. Then for every $A \in \mathcal{M}$ we have, $f_\mathcal{S}^{-1}(f_\mathcal{S}(A)) = A$.

Define $Z'_2 = Z_1 \times \prod \{Q_\alpha | \alpha \in \Sigma'\} \times \prod \{Q_\mathcal{S} = Q | \mathcal{S} \in \mathcal{S}'_\mathcal{N}\}$. Put $f'_2 = f_1 \times (g_\alpha)_{\alpha \in \Sigma'} \times (f_\mathcal{S})_{\mathcal{S} \in \mathcal{S}'_\mathcal{N}} : X \rightarrow Z'_2$. Let $Z_2 = f'_2(X) \subset Z'_2$ and set $f_2 = f'_2 : X \rightarrow Z_2$. Denote $p : Z_2 \rightarrow Z_1$ to be the restriction of the projection of Z_2 to its first coordinate. Surely f_2 is surjective and $p \circ f_2 = f_1$.

We have to show that (f_1, f_2, p) is lifting for simplicial complexes relative to D . Let A be closed in Z_1 , N be a simplicial complex and $h : A \rightarrow |N|$, $F : D \rightarrow |N|$ be maps having the property that $F(x) = h(f_1(x))$ for all $x \in D \cap f_1^{-1}(A)$. If

necessary, replace N by a finite subcomplex whose polyhedron contains the image of F . Extend h to a map $h_0 : G \rightarrow |N|$ where G is a closed neighborhood of A in Z_1 . By virtue of Lemma 2.3, we may assume that $F|G_0 : G_0 \rightarrow |N|$ is homotopic to $h_0 \circ f_1|G_0 : G_0 \rightarrow |N|$, where $G_0 = D \cap f_1^{-1}(G)$.

The family \mathcal{F} is dense, so there exist $\alpha \in \Sigma$ with $A \subset A_\alpha \subset G$ and a simplicial isomorphism $\Phi_\alpha : N \rightarrow N_\alpha$ such that $\Phi_\alpha \circ h_0|A_\alpha \simeq h_\alpha$ as maps of A_α to $|N_\alpha|$. Let $A_0 = D \cap f_1^{-1}(A_\alpha) \subset G_0$. Then $F|A_0 : A_0 \rightarrow |N|$ is homotopic to $h_0 \circ f_1|A_0 : A_0 \rightarrow |N|$. Since F is defined on all of D , then $h_0 \circ f_1|A_0$ extends to a map $\tilde{F} : D \rightarrow |N|$.

The map $\Phi_\alpha \circ \tilde{F}|A_0 = \Phi_\alpha \circ h_0 \circ f_1|A_0 : A_0 \rightarrow |N_\alpha|$ is homotopic to $h_\alpha \circ f_1|A_0 : A_0 \rightarrow |N_\alpha|$. Since $\Phi_\alpha \circ \tilde{F}|A_0$ extends to a map of D , so does $h_\alpha \circ f_1|A_0$. This shows that $\alpha \in \Sigma'$, and $h_\alpha \circ f_1|A_0$ extends to the map $g_\alpha|D : D \rightarrow |N_\alpha|$.

Let $E = p^{-1}(A) \cap f_2(D)$. Define $G_\alpha : Z_2 \rightarrow |N_\alpha|$ to be the restriction of the α -coordinate projection of Z'_2 . Suppose that $z \in E$; then there exists $t \in D$ such that $f'_2(t) = f_2(t) = z$. Then of course $p(z) = p(f_2(t)) = f_1(t)$, so $t \in D \cap f_1^{-1}(A) \subset A_0$. This shows that $h_\alpha(p(z)) = h_\alpha(f_1(t)) = g_\alpha(t)$. On the other hand, $z = f_2(t) = (f_1(t), (g_\alpha(t))_{\alpha \in \Sigma'}, (f_S(t))_{S \in \mathcal{S}'_{\mathcal{N}}})$ so $G_\alpha(z) = g_\alpha(t) = h_\alpha(p(z))$.

This helps one see that $\Phi_\alpha^{-1} \circ h_\alpha \circ p|E : E \rightarrow |N|$ extends to a map of $f_2(D)$ to $|N|$. The last step of this part of the proof is to use the fact that $\Phi_\alpha^{-1} \circ h_\alpha \circ p|E : E \rightarrow |N|$ is homotopic to $h \circ p|E : E \rightarrow |N|$, so the desired map $H : f_2(D) \rightarrow |N|$ extending the latter exists.

Lastly we are to show that (f_1, f_2, p) is C-lifting relative to D . Let \mathcal{S}' be a sequence of open covers of Z_1 such that $f_1^{-1}(\mathcal{S}')|D$ has a C-refinement in D . It is not difficult to see that there is an element \mathcal{S} of $\mathcal{S}'_{\mathcal{N}}$ so that each element of \mathcal{S} refines the corresponding element of \mathcal{S}' and $f_1^{-1}(\mathcal{S})|D$ has a C-refinement in D . Then as provided above, $\mathcal{C}_\mathcal{S}$ is such a C-refinement and if $\mathcal{M} = \bigcup \mathcal{C}_\mathcal{S}$, then $f_S^{-1}(f_S(A)) = A$ for all nonempty $A \in \mathcal{M}$.

It will be sufficient to show that $p^{-1}(\mathcal{S})|f_2(D)$ has a C-refinement in $f_2(D)$. We claim that $f_2(\mathcal{C}_\mathcal{S})$ satisfies this requirement.

First, if $B \in \bigcup \mathcal{S}$, then the surjectivity of f_2 and the fact that $p \circ f_2 = f_1$ can be used to show that $p^{-1}(B) = f_2(f_1^{-1}(B))$. Hence if $A \in \mathcal{M}$ and $A \subset f_1^{-1}(B)$, then $f_2(A) \subset p^{-1}(B)$. This takes care of the refinement part. If $A' \in \mathcal{M}$ and $A \cap A' = \emptyset$, then we claim that $f_2(A) \cap f_2(A') = \emptyset$. To see this, suppose that $a \in A$, $a' \in A'$ and $f_2(a) = f_2(a')$. Then $f_S(a) = f_S(a')$, so $a' \in f_S^{-1}(f_S(A)) \cap A' = A \cap A' = \emptyset$, a contradiction. This proves the necessary discreteness. \square

For completeness, let us state the generalizations of the (expansion) Theorems 1 and 3 of [Ma1] (resp., Theorems 5 and 6 of [Ma2]). As indicated in 4.2 of [Ma2] (and see the limit theorem of [Ru]), these follow from what we have done here and from the techniques of [Ma1].

2.7. Theorem. *Let X be a compact Hausdorff space. Then there exists an inverse system $\mathbf{Q} = (Q_b, q_{b'b}, B)$ of metrizable compacta Q_b with $\text{card}(B) \leq w(X)$ and $\lim \mathbf{Q}$ homeomorphic to X . If K is a CW-complex and $X\tau K$, then $Q_b\tau K$ for each $b \in B$. \square*

2.8. Theorem. *Let X be a compact Hausdorff space. Then there exists a well-ordered inverse system $\mathbf{Q} = (Q_\beta, q_{\beta\beta'}, \beta < \alpha)$, $\alpha = wX$, where X is homeomorphic to $\lim \mathbf{Q}$ and for each $\beta < \alpha$, $wQ_\beta \leq \beta < \alpha$, and $Q_\beta = \lim(Q_\tau, q_{\tau\tau'}, \tau < \beta)$ if β is a limit ordinal (in that case, $q_{\tau\beta}$ is the corresponding projection). If K is a CW-complex and $X\tau K$, then $Q_\beta\tau K$ for each $\beta < \alpha$. \square*

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