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NOT EVERY Q-SET IS PERFECTLY MEAGER IN THE TRANSITIVE SENSE

ANDRZEJ NOWIK AND TOMASZ WEISS

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Abstract. We prove the following theorems:

1. It is consistent with ZFC that there exists a Q - set which is not perfectly meager in the transitive sense.
2. Every set which is perfectly meager in the transitive sense has the $\overline{AFC}$ property.
3. The product of two sets perfectly meager in the transitive sense has also that property.

In this part we prove that it is consistent with ZFC that there are uncountable Q - sets which are not perfectly meager in the transitive sense. Most of the lemmas needed to show this result are based on I. Recław [R], and H. Judah, S. Shelah [JS]. Throughout the proof, terminology and notations from the two papers mentioned above are being used.

We use the following definition (see [NSW]) of sets that are perfectly meager in the transitive sense.

Definition 1. Let X be a subset of the real line (or 2^ω, respectively). We say that X is an AFC' set (perfectly meager in the transitive sense) iff for every perfect set $D \subseteq \mathbf{R}$ ($D \subseteq 2^\omega$, respectively), one can find F, an F_σ set containing X, such that for every $t \in \mathbf{R}$ (2^ω), $(F + t) \cap D$ is meager in the relative topology of D.

In [NSW], it is shown that (assuming Martin's Axiom) the class AFC' is strictly included in the class AFC of perfectly meager sets.

Let us also recall that a set $X \subseteq \mathbf{R}$ (or 2^ω, respectively) is called a Q - set iff its every subset is an F_σ set in the relative topology of X. It is well - known that every Q - set is perfectly meager (see for example [M]).

For A, B, subsets of $\mathbf{R}$ (2^ω), we define $A - B = \{a - b : a \in A, b \in B\}$.

Lemma 1 (Recław). *Suppose that C, D are compact subsets of $\mathbf{R}$ (2^ω) with the property that $(C - C) \cap (D - D) = \{0\}$. Assume that $X \subseteq C$. Then for every function $d : X \to D$, the set $Y_d = \{y_x : x \in X\}$, where $y_x = d(x) + x$, $x \in X$, is a continuous one-to-one preimage of X.*

Proof. See Lemma 1 in [R]. □

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Definition 2. We call a family $\mathcal{A}$ an AFC' - cofinal if for every perfect set D and every F_σ - set F with the property that for any $t \in \mathbf{R}$ (2^ω), $(F+t) \cap D$ is meager in the relative topology of D, there is in $\mathcal{A}$ an F_σ - set F' containing F, so that for each $t \in \mathbf{R}$ (2^ω), $(F'+t) \cap D$ is meager in the relative topology of D.

Lemma 2. *Suppose that in* $\mathbf{R}$ (2^ω) *there is a* Q *- set of cardinality* ω_1 *and that there is an* AFC' *- cofinal family* $\mathcal{A}$ *of cardinality* ω_1*. Then there exists a* Q *- set of cardinality* ω_1 *which is not a member of* AFC'*.*

Proof. We use analogous arguments to Theorem 2 from [R]. Suppose that C, D are disjoint, perfect (compact) subsets of $\mathbf{R}$ that are linearly independent over the rationals [vN]. Without loss of generality we may assume that there exists a Q - set X of cardinality ω_1 included in C. Choose a family $\mathcal{A}$ and let $\{A_x\}_{x\in X}$ be its enumeration. For each $x \in X$ pick $y_x \in D + x$ satisfying $y_x \notin A_x$ if this is possible. If not, let y_x be any element of $D + x$. Clearly, $Y = \{y_x : x \in X\}$ is a Q - set and if $Y \subseteq A_x$ for some $x \in X$, then $D \subseteq A_x - x$ which proves that Y is not an AFC' set. To prove Lemma 2 for subsets of 2^ω, we need the following Claim.

Claim 1. There are C, D disjoint perfect subsets of 2^ω such that $(C-C)\cap(D-D) = \{0\}$.

Proof. We construct trees T, T' by induction. Suppose that $T_n, T'_n \subseteq \{s : s \in 2^n\}$ are given. Choose any $k \geq 3$ and different $t_1, t_2, t'_1, t'_2 \in 2^k$ such that $t_1+t_2 \neq t'_1+t'_2$. Then put $T_{n+1} = \{s^\frown t_1, s^\frown t_2 : s \in T_n\}$ and $T'_{n+1} = \{s^\frown t'_1, s^\frown t'_2 : s \in T'_n\}$. Clearly, $C = [T]$, $D = [T']$ do the job. □

From now on we assume that all sets we deal with are included in 2^ω and for $s \in 2^{<\omega}$ we define a basic clopen set $[s] = \{x \in 2^\omega : s \subseteq x\}$.

Lemma 3. *Assume that* F *is a closed set with the property that for any* $t \in 2^\omega$, $[s] \setminus (F+t) \neq \emptyset$*. Then there exists a clopen set* F' *containing* F *such that for each* $t \in 2^\omega$, $(F'+t)$ *leaves* $[s]$ *uncovered.*

Proof. For each $t \in 2^\omega$ find an open U_t with $t \in U_t$ and F_t, a clopen set containing F, so that $[s] \setminus (U_t + F_t) \neq \emptyset$. Let $t_1, \ldots, t_k$ be such that $2^\omega \subseteq U_{t_1} \cup \ldots \cup U_{t_k}$. Put $F' = F_{t_1} \cap \ldots \cap F_{t_k}$. □

Corollary 1. *Suppose that* F *is a closed set such that for every* $t \in 2^\omega$ *and every* $r \in 2^{<\omega}$, $[r] \setminus (F+t) \neq \emptyset$*. Assume that* F_1 *is a closed set with the property that for any* $t \in 2^\omega$, $[s_0] \setminus (F_1+t) \neq \emptyset$*, where* s_0 *is a fixed element of* $2^{<\omega}$*. Then there is a clopen set* F' *containing* $F \cup F_1$ *such that for any* $t \in 2^\omega$, $[s_0] \setminus (F'+t) \neq \emptyset$*.*

Proof. Apply Lemma 3. □

To prove the main theorem of this part we recall the following notion of forcing introduced by Judah and Shelah in [JS].

Definition 3. We say that $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is a suitable sequence of infinite subsets of ω if and only if:

1) $A_i \in [\omega]^\omega$ for each $i < \omega_1$,
2) $A_i \subseteq_* A_j$ for $i < j < \omega_1$, that is $|A_i \setminus A_j| < \omega$,
3) $a_i \in [A_{i+1} \setminus A_i]^\omega$ for every $i < \omega_1$.

Definition 4. If $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ is a suitable sequence and $X \subseteq \omega_1$, we define the notion of forcing $P(\bar{A}, X)$ as follows: $h \in P(\bar{A}, X)$ if and only if

1) $h : v \to \{0, 1\}$, $v \subseteq \omega$,
2) $\mathrm{dom}(h) \subseteq_* A_i$ for some $i < \omega_1$,
3) for every $j < i_0$, where i_0 is the minimal ordinal (depending on h) such that $\mathrm{dom}(h) \subseteq_* A_{i_0}$, we have that
$$a_j \subseteq_* \mathrm{dom}(h)$$
and
$$j \notin X \text{ if and only if } a_j \subseteq_* h^{-1}(\{0\}),$$
$$j \in X \text{ if and only if } a_j \subseteq_* h^{-1}(\{1\}).$$

We order $P(\bar{A}, X)$ by the reverse inclusion.

In the following lemma we identify elements of $[\omega]^\omega$ with their characteristic functions.

Lemma 4. *Suppose that M is a model of ZFC and G is an M - generic filter in $P(\bar{A}, X)$. Then in $M[G]$ we have that $\{a_i : i \in X\}$ is a relative F_σ subset of the set $\{a_i : i < \omega_1\}$.*

Proof. Notice that $g = \bigcup\{h : h \in G\}$ is a function with $\mathrm{dom}(g) \subseteq \omega$. Also, for each $\alpha < \beta < \omega_1$ and any $h \in P(\bar{A}, X)$ such that α is the minimal ordinal with $\mathrm{dom}(h) \subseteq_* A_\alpha$, we can find $h' \supseteq h$, $h' \in P(\bar{A}, X)$, so that β is the minimal ordinal with $\mathrm{dom}(h') \subseteq_* A_\beta$ (see Lemma 1.3 of [JS]). Thus, by the standard density argument, $i \in X$ if and only if $a_i \subseteq_* g^{-1}(\{1\})$. Clearly, $\{a_i : i \in X\} = (\bigcup_{n\in\omega} \bigcap_{k\geq n,\ k\in\omega\setminus g^{-1}(\{1\})} F_k) \cap \{a_i : i < \omega_1\}$, where $F_k = \{x \in 2^\omega : x(k) = 0\}$. $\square$

Now let $P_0 = \{\langle a_i, A_i : i < j \rangle : j < \omega_1\}$, where each sequence $\langle a_i, A_i : \ i < j \rangle$ satisfies conditions 1) – 3) from Definition 3. We order P_0 by the reverse inclusion.

Definition 5. Let $P_{\omega_2} = \langle P_i, \dot{P}_i : i < \omega_2 \rangle$ be the countable support iteration of forcings, so that:

1) If $0 < i < \omega_2$, then
$$\mathbf{1}_{P_i} \Vdash \dot{X} \subseteq \omega_1 \text{ and } \dot{P}_i = P(\dot{\bar{A}}, \dot{X}),$$
where $\dot{\bar{A}}$ is a P_0-name for a suitable sequence added by a generic filter in P_0.
2) If $i < \omega_2$ and
$$\mathbf{1}_{P_i} \Vdash \dot{X} \subseteq \omega_1,$$
then for some $j \geq i$
$$\mathbf{1}_{P_j} \Vdash \dot{P}_j = P(\dot{\bar{A}}, \dot{X}).$$

Theorem 1. *Suppose that M is a model of ZFC + GCH and let G be an M - generic filter in P_{ω_2}. Then in $M[G]$ we have that there exists a Q - set which is not an AFC′ set.*

Proof. Let $\bar{A} = \langle a_i, A_i : i < \omega_1 \rangle$ be a suitable sequence added by G_0, an M - generic filter in P_0. Clearly, $\{a_i : i < \omega_1\}$ is a Q - set in 2^ω (see Lemma 4). The forcing P_{ω_2} is proper, hence ω_1 is preserved (see Theorem 1.6 (b) of [JS]). Thus, it suffices to show that in $M[G]$ there exists a family $\mathcal{A}$ defined in Lemma 2. By the standard fusion argument we may assume that we essentially use a two-step iteration of the form $P_0 * P(\dot{\bar{A}}, \dot{X})$, where

$$\mathbf{1}_{P_0} \Vdash \dot{X} \subseteq \omega_1$$

and $P(\dot{\bar{A}}, \dot{X})$ is a name for a notion of forcing as in Definition 4 (see the proof of Theorem 1.6 (b) in [JS]).

Claim 2. Let $[s]$ be a (relative) basic clopen set in a perfect set D. Given a condition $p \in P_0 * P(\dot{\bar{A}}, \dot{X})$ of the form $p = p(0) * h$ and a finite set $u \subseteq \omega$ with the property that

$$p(0) \Vdash \mathrm{dom}(h) \cap u = \emptyset$$

and such that

$$p \Vdash \dot{F} \text{ is closed and for any } t \text{ and every } r \in 2^{<\omega}, [r] \setminus (\dot{F} + t) \neq \emptyset,$$

we can find a closed set $F' \in M$ and a condition $p' \leq p$ of the form $p' = p'(0) * h'$ with $p'(0) \Vdash h' \cap u = \emptyset$, so that

$$p' \Vdash \dot{F} \subseteq F' \text{ and for every } t, \quad [s] \setminus (F' + t) \neq \emptyset.$$

Proof. Let $\{u_n\}_{n \leq 2^{|u|}}$ be an enumeration of all 0–1 functions with the domain equal to u. By induction define a sequence $p_{n+1} \leq p_n$, $p_n = p_n(0) * h_n$, $p_{n+1}(0) \Vdash h_{n+1} \leq h_n$ for $n \leq 2^{|n|}$ with the properties (use absoluteness of Corollary 1 relativised to D):

1. $p_n(0) \Vdash \mathrm{dom}(h_n) \cap u = \emptyset$,
2. $$p_n(0) * (h_n \cup u_n) \Vdash \dot{F} \subseteq F_n \text{ and for any } t, \quad [s] \setminus (F_n + t) \neq \emptyset,$$

   where each F_n is a clopen set and $F_{n-1} \subseteq F_n$.

Then put $p' = p_k(0) * h_k$ with $k = 2^{|u|}$ and define $F' = F_k$. □

Claim 3. Suppose that D is a perfect set and

$$p \Vdash \dot{F} \text{ is closed and for every } t, \quad (\dot{F} + t) \text{ is nowhere dense}$$

$$\text{in the relative topology of } D.$$

Then there exists a closed set F', $F' \in M$, and a condition $p' \leq p$, so that

$$p' \Vdash \dot{F} \subseteq F' \text{ and for each } t, \quad (F' + t) \cap D \text{ is nowhere dense}$$

$$\text{in the relative topology of } D.$$

Proof. Let $[s_n]_{n \in \omega}$ be an enumeration of basic clopen sets in D. For every $n \in \omega$, use Claim 2 to define inductively $p_n, u_n, F'_n \in M$ satisfying:

1. $p_{n+1} \leq p_n$, $p_n = p_n(0) * h_n$, $p_n(0) = \langle a^n_j, A^n_j : j < j_n < \omega_1 \rangle$,
2. $$p_n \Vdash \dot{F} \subseteq F'_n \text{ and for every } t, \quad [s_n] \setminus (F'_n + t) \neq \emptyset,$$
3. $u_n \subset u_{n+1}$, $u_{n+1} \setminus u_n \neq \emptyset$, is a sequence of finite subsets of ω and

   $$p_n(0) \Vdash \mathrm{dom}(h_n) \cap u_n = \emptyset.$$

Let $\langle a_j, A_j : j < \sup_{n\in\omega} j_n\rangle$ be such that for any $n \in \omega$, $p_n(0)$ is an initial segment of $\langle a_j, A_j : j < \sup_{n\in\omega} j_n\rangle$ and suppose that $A_\infty = \omega \setminus \bigcup_{n\in\omega} u_n$. Then put $F' = \bigcap_{n\in\omega} F'_n$, $p'(0) = \langle a_j, A_j : j < \sup_{n\in\omega} j_n\rangle \cup \langle A_\infty\rangle$ and define $p' = p'(0) * \bigcup_{n\in\omega} h_n$. Clearly, $p' \leq p$ and

$$p' \Vdash \dot{F} \subseteq \bigcap_{n\in\omega} F'_n \text{ and for every } t, \quad (\bigcap_{n\in\omega} F'_n + t) \cap D$$

is nowhere dense in the relative topology of D.

□

Claim 4. Assume that D is a perfect set and

$$p \Vdash \dot{F} \text{ is an } F_\sigma \text{ - set such that for every } t, (\dot{F} + t) \cap D \text{ is meager}$$

in the relative topology of D.

Then there is $p' \leq p$ and F', an F_σ - set coded in M, with the property that

$$p' \Vdash \dot{F} \subseteq F' \text{ and for any } t, \quad (F' + t) \cap D \text{ is meager}$$

in the relative topology of D.

Proof. Straightforward application of Claim 3. □

Proof of Theorem 1. Notice that by Claim 4 the family

$$\mathcal{A} = \{F : \ F \text{ is a meager } \ F_\sigma \text{ –set coded in } M\}$$

satisfies the assumptions of Lemma 2. □

Following [G1] and [G2] we define the class $\overline{AFC}$.

Definition 6. $A \in \overline{AFC}$ iff for every set $B \subseteq 2^\omega$ for which there exists a 1-1 Borel measurable function $f : B \to A$, we have that $B \in AFC$.

We need the following characterization (see [G2] Lemma 1).

Lemma 5. *$A \in \overline{AFC}$ iff for every set $B \subseteq 2^\omega$ for which there exists a 1-1 continuous function $f : B \to A$, we have that $B \in \mathcal{MGR}$ (meager sets).*

In the next part we will prove the following theorem.

Theorem 2. *Every AFC' set belongs to the class $\overline{AFC}$.*

Proof. Before giving a proof of this theorem, we shall formulate the following auxiliary characterization of the AFC' property.

Lemma 6. *Let $X \subseteq 2^\omega$. The following conditions are equivalent:*

1. *$X \notin AFC'$.*
2. *There exists a sequence $\{Q_n\}_{n<\omega}$ of perfect subsets of 2^ω such that if $X \subseteq \bigcup_{n<\omega} F_n$, $F_n = \overline{F_n}$, then there exist $n, m < \omega$, $t \in 2^\omega$ such that $Q_m + t \subseteq F_n$.*

Proof. 1. $\Rightarrow$ 2. Let $P \subseteq 2^\omega$ be a perfect set for which X does not satisfy conditions from the definition of an AFC' set. Then we put $\{Q_n\}_{n<\omega}$ to be equal to the clopen base of P.

Claim 5. If $\{Q_n\}_{n<\omega}$ is a sequence of perfect sets from 2^ω, then there exists a perfect set $P \subseteq 2^\omega$ such that

$$\forall_{n<\omega}\exists_t\ (Q_n+t)\cap P \text{ has a nonempty interior relative to } P.$$

Proof. Obvious. □

To prove 2. ⇒ 1. apply Claim 5 to the sequence $\{Q_n\}_{n<\omega}$. □

So let $X \in AFC'$, $X \subseteq 2^\omega$, $Y \subseteq 2^\omega$, $f : Y \to X$ be 1 - 1 and continuous. To obtain a contradiction, assume that $Y \notin \mathcal{MGR}$. Let $\{C_n\}_{n<\omega}$ be a clopen base of 2^ω. Put

$$\Lambda = \{n < \omega : |C_n \cap Y| \geq \omega_1\}.$$

For every $m \in \Lambda$ choose a perfect set

$$Q_m \subseteq \overline{f[C_m \cap Y]}.$$

Let $X \subseteq \bigcup_{n<\omega} F_n$, $\overline{F_n} = F_n$. Then, by Lemma 6,

$$\forall_{m\in\Lambda}\forall_{n\in\omega}\forall_{t\in 2^\omega} Q_m + t \not\subseteq F_n. \tag{1}$$

Take a closed $K_n \subseteq 2^\omega$ such that

$$f^{-1}(F_n) = K_n \cap Y.$$

Consider two cases:
Case 1.

$$\text{Suppose that } \forall_{n<\omega}|int(K_n)\cap Y| \leq \omega.$$

Then $Y \subseteq \bigcup_{n<\omega}[int(K_n)\cap Y] \cup [K_n \setminus int(K_n)] \in \mathcal{MGR}$.

Case 2.

$$\text{Assume that } \exists_{n_0\in\omega}|int(K_{n_0})\cap Y| \geq \omega_1.$$

Choose $m_0 < \omega$ such that $|C_{m_0}\cap Y| \geq \omega_1$ and $C_{m_0} \subseteq int(K_{n_0})$.
Now $m_0 \in \Lambda$, thus

$$Q_{m_0} \subseteq \overline{f[C_{m_0}\cap Y]} \subseteq \overline{ff^{-1}(F_{n_0})} \subseteq \overline{F_{n_0}} = F_{n_0}.$$

However this is a contradiction with (1). □

Using this theorem and results from [NSW], we obtain several interesting conclusions.

Conclusion 1. Every strongly first category subset of 2^ω is an $\overline{AFC}$ set.

Proof. From [NSW] we know that every strongly first category set is an AFC' set. □

Conclusion 2. Assume that $X \subseteq 2^\omega$ is a strongly first category set, $f : Y \to 2^\omega$ is a Borel one-to-one function. Then the preimage $f^{-1}[X]$ belongs to the class of $\overline{AFC}$ sets.

Proof. Every preimage of an $\overline{AFC}$ set by a Borel one-to-one function is an $\overline{AFC}$ set as well. □

Conclusion 3. No uncountable Borel image of a Luzin set can be a strongly first category set.

Proof. Let $L \subseteq 2^\omega$ be a Luzin set and let $b : L \to f[L]$ be a one-to-one Borel function. We may assume that b is defined on 2^ω. We can find a meager set $M \subseteq 2^\omega$ such that $b \restriction 2^\omega \setminus M$ is continuous. Assume that $b[L]$ is a strongly first category set. Using the definition of an $\overline{AFC}$ set, we see that $L \setminus M$ is meager, so L is countable. □

We conflate $2^\omega \times 2^\omega$ with the space 2^ω via the standard homeomorphism. Recall that assuming Martin's Axiom one can find two AFC sets, say $X, Y \subseteq 2^\omega$, such that their product $X \times Y$ is not an AFC set (I. Recław). It is still an open question whether the existence of two such sets can be proved in ZFC only. On the other hand, the sharper class $\overline{AFC}$ has the property that the product of two $\overline{AFC}$ sets is again an $\overline{AFC}$ set (see [Z]). We prove that (without any extra assumptions) the product of two AFC' subsets of 2^ω is an AFC' set, too.

Theorem 3. *The product of two AFC' sets is an AFC' set.*

Proof. We use Lemma 6. To obtain a contradiction assume that one can find a sequence $\{Q_n\}_{n<\omega}$ of perfect sets from $2^\omega \times 2^\omega$ such that for every sequence $\{F_m\}_{m<\omega}$ of closed subsets from $2^\omega \times 2^\omega$ we have

$$\exists_{t \in 2^\omega \times 2^\omega} \exists_{n,m<\omega} Q_n + t \subseteq F_m.$$

Put

$$C = \{n \in \omega : |\pi_x[Q_n]| = \mathfrak{c}\}.$$

For every $n \in C$ let

$$Q_n^x \subseteq \pi_x[Q_n]$$

be any perfect set. For every $n \in \omega \setminus C$ let

$$Q_n^y \subseteq \pi_y[Q_n]$$

also be any perfect set. $X \in AFC'$, so we can find a sequence $\{F_m^x\}_{m\in\omega}$ of closed sets such that

$$X \subseteq \bigcup_{m\in\omega} F_m^x$$

and

$$\forall_{t\in 2^\omega} \forall_{m\in\omega} \forall_{n\in C} Q_n^x + t \nsubseteq F_m^x.$$

Since $Y \in AFC'$, one can find a sequence $\{F_p^y\}_{p\in\omega}$ of closed subsets of 2^ω such that

$$Y \subseteq \bigcup_{p\in\omega} F_p^y$$

and

$$\forall_{t\in 2^\omega} \forall_{p\in\omega} \forall_{n\in\omega\setminus C} Q_n^y + t \nsubseteq F_p^y.$$

Consider a sequence $\{F_m^x \times F_p^y\}_{p,m<\omega}$ of closed subsets of $2^\omega \times 2^\omega$. Since

$$\bigcup_{m,p<\omega} F_m^x \times F_p^y \supseteq X \times Y,$$

we can find $(a, b) \in 2^\omega \times 2^\omega$ and $m_0, n_0, p_0 \in \omega$ such that

$$(a, b) + Q_{n_0} \subseteq F_{m_0}^x \times F_{p_0}^y.$$

Now, if $n_0 \in C$, we have that

$$a + Q^x_{n_0} \subseteq F^x_{m_0},$$

which is a contradiction. So let $n_0 \in \omega \setminus C$, but then

$$Q^y_{n_0} + b \subseteq F^y_{p_0},$$

which is again a contradiction. □

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Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00 – 950 Warsaw, Poland

E-mail address: matan@paula.univ.gda.pl

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warszawa, Poland

Current address: Institute of Mathematics, WSRP, 08-110 Siedlce, Poland

E-mail address: weiss@wsrp.siedlce.pl