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EXOTIC SMOOTH STRUCTURES ON $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$

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ABSTRACT. We construct exotic $3\mathbf{CP}^2\#10\overline{\mathbf{CP}}^2$ and $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$ as a corollary of recent results of I. Dolgachev and C. Werner concerning a numerical Godeaux surface. We also construct another exotic $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$ using the surgery techniques of R. Fintushel and R. J. Stern. We show that these 4-manifolds are irreducible by computing their Seiberg-Witten invariants.

1. Introduction

It is an intriguing and very tricky problem in differential topology to construct exotic smooth 4-manifolds with small Euler characteristics. For example, given an ordered pair of positive integers (m, n), we can ask whether there exists a smooth closed simply-connected 4-manifold that is homeomorphic but not diffeomorphic to $m\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$. Note that such an exotic 4-manifold has $e = m + n + 2$, $b_2^+ = m$, and $b_2^- = n$. Probably the most famous example of this kind is the Barlow surface, which corresponds to the pair $(1, 8)$ (cf. [B] and [Ko]). The Barlow surface has the smallest Euler characteristic among all known closed oriented simply-connected 4-manifolds with more than one smooth structure. In this paper we construct three exotic 4-manifolds corresponding to the pairs $(3, 10)$ and $(3, 12)$. Namely, we prove the following

Theorem 1.1. *There exists a smooth closed simply-connected irreducible symplectic 4-manifold X_n that is homeomorphic but not diffeomorphic to $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ for $n = 10, 12$.*

For the $n = 12$ case, we construct two seemingly very different exotic manifolds, denoted M and P. The author does not know whether these two exotic manifolds are in fact diffeomorphic. We simplify the notation by letting $X = X_{10}$. In the sequel [P] we will construct other exotic 4-manifolds corresponding to the pairs $(3, 11)$ and $(3, 13)$. To the author's best knowledge, X is the "smallest" known example of an exotic 4-manifold for the case $b_2^+ > 1$. Along this vein, the author presently knows of no example of a smooth oriented closed simply-connected 4-manifold Y with $b_2^+(Y) = 3$ and $b_2^-(Y) < 10$ that is not already diffeomorphic to some standard $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ or $3(S^2 \times S^2)$. To get a feel for the relative size of these Betti numbers, we note that the celebrated $K3$ surface satisfies $b_2^+(K3) = 3$, $b_2^-(K3) = 19$. (Of course, the $K3$ surface has an even intersection form and hence is

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not homotopy equivalent to $3\mathbf{CP}^2\#19\overline{\mathbf{CP}}^2$.) In the fundamental paper [G], Gompf constructed symplectic 4-manifolds with $b_2^+ = 3$, $14 \leq b_2^- \leq 18$. Later on those 4-manifolds were studied by Stipsicz, Szabó and Yu (cf. [St], [Sz1], [Y]). Using Donaldson's polynomial invariants, they proved that many of Gompf's examples are in fact irreducible, and in particular are not diffeomorphic to $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$.

Each of our manifolds X and P is going to be a symplectic sum of two symplectic manifolds, one of which is the numerical Godeaux surface discovered by Craighero and Gattazzo; cf. [CG]. A crucial step in the construction is the result of Dolgachev and Werner in [DW] that this complex surface admits a genus 2 fibration over the 2-sphere.

Our last example M is obtained by replacing the numerical Godeaux surface in the construction of X with a homotopy rational elliptic surface of Fintushel and Stern in [FS2]. The details of the constructions are to be found in Sections 2 and 4. In Sections 3 and 5, we compute the Seiberg-Witten invariants of our 4-manifolds and use this information to conclude that our manifolds are irreducible.

Lastly it remains an interesting open problem in the case when $b_2^+ = 3$ as to whether one can realize even smaller values of b_2^- and thus "beat" the example X in this paper, or even more ambitiously, the Barlow surface in the $b_2^+ = 1$ case.

2. Construction of X

We assume that the reader is familiar with the rudiments of the Seiberg-Witten monopole theory on symplectic 4-manifolds. For more on the basics of Seiberg-Witten invariants, see [KM], [M], and [Wi]. We summarize one of the standard results needed in the following proposition.

Proposition 2.1 (See [Ta]). *Let X be a closed symplectic 4-manifold and suppose $b_2^+(X) > 1$. Then X does not admit any Riemannian metric of positive scalar curvature.* □

We first review some properties of the numerical Godeaux surface S of Craighero and Gattazzo (cf. [DW]). S is a simply-connected minimal complex surface of general type with $q = p_g = 0$ and $K^2 = 1$. S is homeomorphic to $\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$ and the Barlow surface. It is not yet known whether S is deformation equivalent to the Barlow surface, and it is an open problem whether they are diffeomorphic or not. We have $sign(S) = -7$ and $e(S) = 11$. There is a genus 2 fibration $\pi : S \to \mathbf{CP}^1$. Denote the regular fiber by Σ_2. Note that, as the fiber, Σ_2 has self-intersection 0. Recall that S contains four distinct tori E_j $(j = 1, \dots, 4)$ with self-intersection (-1). Now there exists a class C such that $\Sigma_2 \cdot C = 1$ inside S. For example, we can take $C = E_1 - K_S$, where K_S is the canonical class. It follows that the fibration π has no multiple fibers. After a small perturbation we can assume that π is a Lefschetz fibration (cf. [L] and [GS], Chapter 8). Furthermore we can assume that the vanishing cycle on the fiber Σ_2 is homologically nontrivial.

Our second building block will be the fiber sum of two copies of the Kodaira-Thurston manifold (cf. [G], p. 570). Let (a, b) be the generators of $\pi_1(T^2)$, and $\varphi : T^2 \to T^2$ a diffeomorphism which maps (a, b) to (ab, b). Let Y be the mapping torus of φ, i.e., a T^2 bundle over S^1 with monodromy φ. The Kodaira-Thurston manifold Z is defined to be the product $S^1 \times Y$. Z fibers over T^2 with T^2 fiber, with trivial monodromy around the first factor, and Z has a section of square 0. Now take two copies Z and Z', and fix fibers $F \subset Z$ and $F' \subset Z'$. Let $f : F \to F'$ be a diffeomorphism which maps (a, b) into $(-b', a')$. Let W denote the induced fiber sum $Z\sharp_f Z'$. Note that W fibers over a genus 2 surface, and the fibration admits a section Σ of square 0, which is obtained by pasting together the sections of Z and Z'. Now W has a symplectic structure, and Σ is a symplectic submanifold (cf. [Th] and [G]). Note that there is an element $g \in \pi_1(\Sigma)$ such that the monodromy relation $g^{-1}ag = ab$ holds inside $\pi_1(W)$.

Now we have all the ingredients needed for our construction. Our manifold X will be the symplectic sum of W and S along Σ and Σ_2. For more on the details of the symplectic sum operation, we refer to [G].

We identify the tubular neighborhoods N_1 of Σ and N_2 of Σ_2 via a diffeomorphism $\psi : N_1 - \Sigma \to N_2 - \Sigma_2$, which preserves the orientations on the normal disks. We choose the gluing map ψ in such a way that ψ maps the generator g to the vanishing cycle on the fiber Σ_2 in S. Let

$$X = W\#_\psi S = (W - \Sigma) \cup_\psi (S - \Sigma_2),$$

where we use ψ to identify $N_1 - \Sigma$ and $N_2 - \Sigma_2$. X is a smooth, closed, oriented 4-manifold. By [G], X admits a canonical symplectic structure, which is obtained from the corresponding symplectic structures of W and S.

Lemma 2.2. *X is simply-connected.*

Proof. Let μ be a circle which represents the meridian of Σ_2, i.e. the boundary of a normal disk at some point of Σ_2. Since S is simply-connected, an easy application of van Kampen's theorem says that $\pi_1(S - \Sigma_2)/\langle\mu\rangle = 1$. Now note that $\mu = [a, b]$ in $\pi_1(W\#_\psi S)$. But in the group $\pi_1(W\#_\psi S)$ we have $g = 1$ which implies that $b = 1$, which in turn implies that $\mu = 1$. We conclude that the homomorphism, $\pi_1(S-\Sigma_2) \to \pi_1(W\#_\psi S)$, induced by the inclusion map, is the zero homomorphism.

It can be shown (cf. [G], p.571) that $\pi_1(W - \Sigma)/\langle\pi_1(\Sigma^{\|})\rangle = 1$, where $\Sigma^{\|}$ is a parallel copy of Σ in $(W - \Sigma)$. Since Σ gets identified with Σ_2 in S and the composition of inclusions, $\Sigma^{\|} \hookrightarrow (S - \Sigma_2) \hookrightarrow X$, induces the zero map on the fundamental groups, an easy application of van Kampen's theorem gives $\pi_1(X) = 1$. □

Lemma 2.3. *Let $X = W\#_\psi S$ be as above. Then $e(X) = 15$ and $sign(X) = -7$.*

Proof. Various topological invariants behave nicely under the symplectic sum operation (cf. [G], p. 535 for the general formulae):

$$sign(W\#_\psi S) = sign(W) + sign(S) = sign(S),$$

$$e(W\#_\psi S) = e(W) + e(S) - 2e(\Sigma) = e(S) + 4.$$

□

Note that $b_2^+(X) = 3$ and $b_2^-(X) = 10$. X has an odd intersection form $3\langle 1\rangle \oplus 10\langle -1\rangle$. According to Freedman's famous theorem (cf. [FQ]), X is homeomorphic to $3\mathbf{CP}^2\#10\overline{\mathbf{CP}}^2$. Now it is well known that $3\mathbf{CP}^2\#10\overline{\mathbf{CP}}^2$ admits a metric of positive scalar curvature (see, e.g. [Sa]); hence by Proposition 2.1 X cannot be diffeomorphic to $3\mathbf{CP}^2\#10\overline{\mathbf{CP}}^2$.

3. Irreducibility and SW-invariants of X

In this section we show that X is irreducible. Recall that a smooth closed simply-connected 4-manifold X is called *irreducible* if each connected sum decomposition of X as $X = Y\#Z$ satisfies that either Y or Z is a homotopy S^4. Irreducibility of X will follow easily from the computation of SW-invariants of X using the product formula of [MST]. In order to use the product formula we must first compute the SW-invariants of each summand. Since $b_2^+(S) = 1$, we must choose one of two chambers of the double cone

$$\{x \in H^2(S;\mathbf{R}) \mid x \cup x > 0\}$$

to define SW-invariants of S. As in [MST], we choose $SW = SW^\omega$, where ω is a symplectic form coming from the Kähler structure on S. Since $b_2^-(S) \leq 9$, there is a unique *metric* chamber for S, and from now on we shall compute the SW-invariants corresponding to that chamber (see Lemma 3.2 of [Sz2]).

Lemma 3.1. *If $SW_S(L) \neq 0$, then $L = \pm K_S$, where K_S is the canonical class of S.*

Proof. Since S is a minimal algebraic surface of general type, the only SW-basic classes of S are $\pm K_S$ (cf. [Wi]). □

Lemma 3.2. *Let F^* denote the Poincaré dual of the homology class of the fiber in W. Then the only possible SW-basic classes of W are $0, \pm 2F^*$.*

Proof. This is an easy application of the generalized adjunction inequality for the SW-basic classes (cf. [OS]), using the fact that $H_2(W;\mathbf{Z}) \cong \mathbf{Z}^6$ (cf. [Sz1], p. 413) has generators consisting of the section Σ and five tori (one of which is the fiber F). □

Now we are ready to prove

Theorem 3.3. *Let K_X denote the canonical class of the symplectic structure on X. Then $SW_X(\pm K_X) = \pm 1$, and $SW_X(L) = 0$ if $L \neq \pm K_X$.*

Proof. The first statement is proved in [Ta]. It remains to prove that there are no other basic classes. As usual, we blur the distinction between cohomology classes and their Poincaré duals. Note that $F \cdot \Sigma = 1$ and $\Sigma_2 \cdot K_S = 2$. Recall from [DW] that there is a smooth rational curve R on S such that $K_S \cdot R = 1$, $R^2 = -3$, and $\Sigma_2 \cdot R = 6$. We can find a connected smooth surface Γ representing K_S. Now we describe how to patch together Γ and two copies of the fiber F inside X. Since the identification $\Sigma = \Sigma_2$ is made in X, we can cut out small disks from both F and Γ, centered at their intersections with Σ and Σ_2 respectively, and then glue together the resulting three open surfaces, Γ^0 and two copies of F^0, along two meridional tubes of $\Sigma = \Sigma_2$ to get a new smooth surface $\overline{\Gamma} = \Gamma\#2F$. Similarly we can form a smooth genus 6 surface $R_6 = R\#6F$ inside X. In X, we have $\overline{\Gamma}^2 = \Gamma^2 = 1$, $R_6^2 = R^2 = -3$, and $\overline{\Gamma} \cdot R_6 = 1$. Now $b_2(X) = 13$ and we easily obtain the following orthogonal decomposition of $H_2(X)$ with respect to the intersection pairing:

$$H_2(X;\mathbf{Z}) \cong \mathcal{T} \oplus \langle \overline{\Gamma}, R_6 \rangle \oplus \mathcal{N}.$$

Here, $\mathcal{T} \cong \mathbf{Z}^4$ comes from $\langle \Sigma, F \rangle^\perp \subset H_2(W)$ and has generators consisting of four tori of self-intersection 0. $\mathcal{N} \cong \mathbf{Z}^7$ is negative definite and comes from $\langle \Sigma_2, \overline{\Gamma} \rangle^\perp \subset$

$H_2(S)$. Note that the generators of $\mathcal{T}$ and $\mathcal{N}$ lie away from the gluing area of symplectic sum operation $\#_\psi$.

Now suppose $SW_X(L) \neq 0$. Then by repeated applications of the generalized adjunction inequality, L is orthogonal to $\mathcal{T}$, i.e.,

$$L = a\overline{\Gamma} + bR_6 + \nu,$$

where $\nu \in \mathcal{N}$, and $|2a + 6b| \leq 2$. But X is symplectic, hence of SW-simple type. This implies that $L^2 = 2e(X) + 3sign(X) = 9$. Since $a^2 + 2ab - 3b^2 \geq L^2$, we must have $a \neq -3b$. Now

$$\langle \pm L, \Sigma \rangle = \pm(2a + 6b) = 2 = 2g(\Sigma) - 2,$$

so we can apply the product formula in [MST]. From the computations in two previous lemmas applied to the product formula, we easily see that L must now be orthogonal to $\mathcal{N}$, i.e. $L = (-3b \pm 1)\overline{\Gamma} + bR_6$. From $L^2 = a^2 + 2ab - 3b^2 = 9$, we conclude that $L = \pm(7\overline{\Gamma} - 2R_6)$. By the pigeonhole principle,

$$L = \pm K_X = \pm(7\overline{\Gamma} - 2R_6).$$

□

Corollary 3.4. *X is irreducible.*

Proof. Since X has nontrivial Seiberg-Witten invariants, it follows that every connected sum decomposition of X as $X = Y\#Z$ satisfies that one of the pieces, say Z, is a homotopy $n\overline{\mathbf{CP}}^2$ with some $n \geq 0$. If $n > 0$, then the blow-up formula for the SW-invariant (cf. [FS1] or [Sa]) implies the existence of SW-basic classes L, L' of X with $(L - L')^2 = -4$. But this is easily seen to be impossible. □

This completes the proof of Theorem 1.1 for the case when $n = 10$.

Remark 3.5. The connected sum theorem for Seiberg-Witten invariant (cf. [Sa]) can be invoked to say that the SW-invariant of $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ is 0. Combined with Theorem 3.3, this gives an alternative proof that X is not diffeomorphic to $3\mathbf{CP}^2\#10\overline{\mathbf{CP}}^2$.

4. Construction of M and P

First we construct M. The construction is very similar to that of X but now we will work with a homotopy rational elliptic surface in place of the numerical Godeaux surface. Recall that $E(1) = \mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$ has a structure of an elliptic fibration over $\mathbf{CP}^1$. That is, there is a holomorphic projection $p : E(1) \to S^2$ with the generic fiber $F = T^2$, a torus. The projection admits a section S, i.e., an embedding $S^2 \to E(1)$, which transversely intersects every fiber exactly once. The normal Euler class of S is -1. (Note that S is just one of the nine exceptional divisors of the blow-ups.) Roughly speaking, our homotopy rational elliptic surface will be obtained from $E(1)$ by removing a tubular neighborhood of a torus fiber F and replacing it with the complement of the trefoil knot in S^3 crossed with S^1. This construction is a special case of the powerful machinery developed by Fintushel and Stern in [FS2].

Let K denote the trefoil knot in S^3 and m a meridional circle to K. Perform the 0-framed surgery on K and call the resulting 3-manifold by $S^3(K)$. In $S^3(K) \times S^1$ we have the smoothly embedded torus $T_m = m \times S^1$ of self-intersection 0. Since a neighborhood of m has a canonical framing in $S^3(K)$, a neighborhood of the torus

T_m in $S^3(K) \times S^1$ has a canonical identification with $T_m \times D^2$. Let $E(1)_K$ denote the fiber sum

$$E(1)\#_{F=T_m}(S^3(K) \times S^1) = [E(1) - (F \times D^2)] \cup_\eta [(S^3(K) \times S^1) - (T_m \times D^2)],$$

where the two pieces are glued together so as to preserve the homology class $\alpha = [pt \times \partial D^2]$. Fintushel and Stern showed that $E(1)_K$ is in fact homotopy equivalent to $E(1)$. Moreover, using Thurston's theorem [Th] and the fact that K is a fibered knot with a punctured torus as a fiber, they showed that $E(1)_K$ is in fact symplectic.

Recall that the Seifert surface Σ^0 of the trefoil knot K is a punctured torus. Since the gluing map η sends the homology class α into the homology class $[K \times pt] = [\partial\Sigma^0 \times pt]$ in $(S^3(K) \times S^1) - (T_m \times D^2)$, we can glue together Σ^0 and the punctured section $S - (F \times D^2)$, and get a smooth torus Σ_1 in $E(1)_K$. Note that $[\Sigma_1] \cdot [\Sigma_1] = [S] \cdot [S] = -1$ and $[F] \cdot [\Sigma_1] = [F] \cdot [S] = 1$. Symplectically blowing up and then down, we resolve the intersection of F and Σ_1, and get a smooth genus 2 surface Σ_2 with self-intersection $[\Sigma_2] \cdot [\Sigma_2] = ([F] + [\Sigma_1]) \cdot ([F] + [\Sigma_1]) = 1$. Gompf's construction tells us that Σ_1 is a symplectic submanifold of $E(1)_K$ (since Σ_1 is the symplectic sum of the symplectic section S in $E(1)$ and the symplectic fiber $\overline{\Sigma^0}$ of $S^3(K) \times S^1$). It follows that Σ_2 is a symplectic surface in $E(1)_K$.

Now by blowing up a point on Σ_2 and taking the proper transform, we get a symplectic genus 2 surface $\overline{\Sigma}_2$ with self-intersection 0 inside $E(1)_K\#\overline{\mathbf{CP}}^2$. We denote $E(1)_K\#\overline{\mathbf{CP}}^2$ by $\overline{R}$. $\overline{R}$ is simply-connected, $e(\overline{R}) = 13$, and $sign(\overline{R}) = -9$.

Let W and Σ be as in Section 2. As before, we symplectically sum W and $\overline{R}$ along Σ and $\overline{\Sigma}_2$, and define

$$M = W\#_\psi\overline{R} = (W - \Sigma) \cup_\psi (\overline{R} - \overline{\Sigma}_2),$$

where ψ is some suitable diffeomorphism as in Section 2.

Lemma 4.1. (i) *M is a smooth, closed, oriented, symplectic 4-manifold.*
(ii) *M is simply-connected.*
(iii) *M is homeomorphic to* $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$.

Proof. Part (i) is immediate. The proofs of the other parts mirror the proofs of corresponding statements for X in Section 2. For (ii), we note that $\pi_1(\overline{R} - \overline{\Sigma}_2) = 1$, and proceed as in Section 2. For (iii), we calculate that $e(M) = 17$ and $sign(M) = -9$. Since M has an odd intersection form and is smooth, Freedman's classification theorem (cf. [FQ]) gives our result. □

Recall that $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$ admits a metric of positive scalar curvature. Hence by Proposition 2.1, M cannot be diffeomorphic to $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$. We have thus proved Theorem 1.1 for the case $n = 12$, except for the irreducibility condition which will be proved in the next section.

Now we proceed with the construction of P. This time around, we keep the numerical Godeaux surface summand S in the construction of X, but replace the W summand with $T^4\#2\overline{\mathbf{CP}}^2$. Choose four distinct points $p_i \in S^1$, $i = 1, \ldots, 4$. Let us define smoothly embedded 2-tori in the 4-torus $T_{12} \subset T^4$, $T_{34} \subset T^4$ by

$$T_{12} = S^1 \times S^1 \times p_1 \times p_2, \;\; T_{34} = p_3 \times p_4 \times S^1 \times S^1.$$

We choose a symplectic form on T^4 for which T_{12}, T_{34} are symplectic submanifolds. Note that $[T_{12}] \cdot [T_{34}] = 1$ and $[T_{12}]^2 = [T_{34}]^2 = 0$. By symplectically resolving the intersection of T_{12} and T_{34}, and then blowing up twice to reduce the self-intersection, we obtain a symplectic genus 2 surface $\tilde{\Sigma} \hookrightarrow T^4\#2\overline{\mathbf{CP}}^2$ with self-intersection 0. We denote $T^4\#2\overline{\mathbf{CP}}^2$ by V. Note that $\pi_1(V) \cong \pi_1(T^4) \cong \mathbf{Z}^4$, $e(V) = 2$, and $sign(V) = -2$.

Let S and Σ_2 be as in Section 2. As before, we symplectically sum V and S along $\tilde{\Sigma}$ and Σ_2, and define

$$P = V\#_\psi S = (V - \tilde{\Sigma}) \cup_\psi (S - \Sigma_2),$$

where ψ is some suitable diffeomorphism as in Section 2.

Lemma 4.2. (i) *P is a smooth, closed, oriented, symplectic 4-manifold.*
(ii) *P is simply-connected.*
(iii) *P is homeomorphic to $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$.*

Proof. Part (i) is immediate. For (ii), we note that if $\tilde{\Sigma}^{\|}$ is a parallel copy of $\tilde{\Sigma}$ in $(V - \tilde{\Sigma})$, then the inclusion induces a surjection $\pi_1(\tilde{\Sigma}^{\|}) \to \pi_1(V - \tilde{\Sigma})$. In particular, the meridian of $\tilde{\Sigma}$ is killed by an embedded disk coming from the exceptional divisor of a blow-up, and hence we can proceed as before. For (iii), we calculate that $e(P) = 17$, $sign(P) = -9$, and then invoke Freedman's theorem. □

For the same reason as before P cannot be diffeomorphic to $3\mathbf{CP}^2\#12\overline{\mathbf{CP}}^2$.

5. Irreducibility and SW-invariants of M and P

We keep the conventions in Section 3 regarding the choice involved in the definition of the SW-invariant for the $b_2^+ = 1$ case. A lot of times we abuse notation and use the same capital letter to denote a surface, its homology class, or the Poincaré dual of its homology class. The proofs in this section will be rather short and sketchy since they are very similar to the proofs of the corresponding statements in Section 3.

Lemma 5.1. *Let T denote the Poincaré dual of the homology class of the fiber in $E(1)$. Then the only SW-basic classes of $E(1)_K$ are $\pm T$.*

Proof. We refer the reader to the last section of [FS2], where the lemma is proved for the more general case when K is an arbitrary twist knot. Note that the trefoil is (-1)-twist knot. □

Lemma 5.2. *The only SW-basic classes of $\overline{R}$ are $\pm(T \pm E)$, where E is Poincaré-dual to the exceptional divisor of the blow-up.*

Proof. Immediate from the blow-up formula for SW-invariants. □

Theorem 5.3. *Let K_M denote the canonical class of the symplectic structure on M. Then $SW_M(\pm K_M) = \pm 1$, and $SW_M(L) = 0$ if $L \neq \pm K_M$.*

Proof. The argument is completely analogous to the proof of Theorem 3.3. We first single out two surfaces in M which intersect $\Sigma = \overline{\Sigma}_2$ transversally at one point.

Let E be the exceptional divisor of $\overline{R}$, and F the fiber of W. Since $E \cdot \overline{\Sigma}_2 = 1 = F \cdot \Sigma$, we can cut out a small disk from both F and E, centered at the intersection with Σ and $\overline{\Sigma}_2$ respectively, and then glue together the resulting open surfaces F^0 and E^0 along a meridional tube of $\Sigma = \overline{\Sigma}_2$ to get a new smooth torus Ξ inside M. We have $\Xi \cdot \Xi = (F + E)^2 = -1$ and $\Xi \cdot \Sigma = (F + E) \cdot \Sigma = 1$.

Let T denote the fiber of $E(1)$. Then $T \cdot \overline{\Sigma}_2 = 1$. As before, we can cut out a small disk from both F and T, centered at the intersection with Σ and $\overline{\Sigma}_2$

respectively, and then glue together the resulting open surfaces F^0 and T^0 along a meridional tube of $\Sigma = \overline{\Sigma}_2$ to get a new smooth genus 2 surface Θ inside M. We have $\Theta \cdot \Theta = (F+T)^2 = 0$, $\Theta \cdot \Sigma = (F+T) \cdot \Sigma = 1$, and $\Theta \cdot \Xi = 0$.

Now suppose $SW_M(L) \neq 0$. As before, we use the generalized adjunction inequality (cf. [OS]) to show that L is orthogonal to the four tori coming from W and has the form

$$L = a\Sigma + b\Xi + c\Theta + \nu,$$

where $|a| \leq 2$, $|a-b| \leq 1$, $|b+c| \leq 2$, and $\langle \nu, \Sigma \rangle = 0$. Note that $\Xi^2 = -1 < 0$, so we have to use the adjunction inequality for *negative* self-intersections [OS] to get the inequality $|a-b| \leq 1$. Since M is of simple type, $L^2 = 2e(M) + 3sign(M) = 7$. This constraint forces $\langle \pm L, \Sigma \rangle = 2g(\Sigma) - 2$, so we can use the product formula of [MST]. Using Lemmas 3.2 and 5.2, we immediately see from the product formula that $\nu = 0$. In the end, we must have

$$L = \pm(2\Sigma + \Xi + \Theta) = \pm K_M.$$

□

Corollary 5.4. *M is irreducible.*

Proof. Exactly the same as the proof of Corollary 3.4. □

This completes the proof of Theorem 1.1 for the case $n = 12$. Finally we turn our attention to P.

Lemma 5.5. *The only SW-basic classes of V are $\pm D_1 \pm D_2$, where D_i are Poincaré-dual to the exceptional divisors of the blow-ups.*

Proof. By the generalized adjunction inequality, 0 is the only SW-basic class of T^4. Now apply the blow-up formula for SW-invariants. □

Theorem 5.6. *Let K_P denote the canonical class of the symplectic structure on P. Then $SW_P(\pm K_P) = \pm 1$, and $SW_P(L) = 0$ if $L \neq \pm K_P$.*

Proof. Completely analogous to the arguments made for X and M. We use the previous cut-and-paste method to make sense out of the expressions $[(D_1 + T_{12}) \# K_S]$, $[(D_2 + T_{12}) \# K_S]$, and $[(D_1 + D_2) \# K_S]$. (Here K_S is the canonical class of S.) If L is a basic class of P, then $L^2 = 7$. From the generalized adjunction inequality [OS] (e.g. applied to smoothly embedded surfaces like Σ_2, $E_1 \# 3T_{12}$ and $R \# 6T_{34}$, etc.) and the product formula [MST], it easily follows that

$$L = \pm(2[\tilde{\Sigma}] + [(D_1 + D_2) \# K_S]) = \pm K_P.$$

□

Corollary 5.7. *P is irreducible.*

Proof. Exactly the same as the proof of Corollary 3.4. □

Remark 5.8. We note that the Seiberg-Witten theory does not provide means to distinguish M and P. Thus it remains an open question whether they are diffeomorphic or not.

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