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EXOTIC SMOOTH STRUCTURES ON $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$, PART II

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Abstract. We construct exotic $3\mathbf{CP}^2\#11\overline{\mathbf{CP}}^2$ and $3\mathbf{CP}^2\#13\overline{\mathbf{CP}}^2$ using the surgery techniques of R. Fintushel and R.J. Stern. We show that these 4-manifolds are irreducible by computing their Seiberg-Witten invariants.

1. Introduction

This paper is a sequel to [P]. For some history and general remarks on distinguishing smooth structures on $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$, we refer to Section 1 of [P]. Our main result is the following

Theorem 1.1. *There exists a smooth closed simply-connected irreducible symplectic 4-manifold X_n that is homeomorphic but not diffeomorphic to $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ for each integer $10 \leq n \leq 13$.*

The even cases $n = 10, 12$ were dealt with in [P]. In this paper we prove the remaining $n = 11, 13$ cases and complete our picture. Together these constructions provide the "smallest" known examples of an exotic closed simply-connected oriented 4-manifold with $b_2^+ > 1$. It remains an open problem whether there is an exotic $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ with $n < 10$.

2. Construction of X_n

We start out by recalling the following

Proposition 2.1 (See [T]). *Let X be a closed symplectic 4-manifold and suppose that $b_2^+(X) > 1$. Then X does not admit any Riemannian metric of positive scalar curvature.* □

Our main building block will be a homotopy rational elliptic surface of Fintushel and Stern in [FS2]. First, let us recall some basic properties of the rational elliptic surface $E(1) = \mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$. Let $B_{2,1} \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ denote the union of four vertical and two horizontal spheres in the direct product $\mathbf{CP}^1 \times \mathbf{CP}^1$. More precisely, we choose six distinct points $p_1, \ldots, p_4, q_1, q_2$ in $\mathbf{CP}^1$ and define the nodal curve

$$B_{2,1} := \bigcup_{i=1}^{4}(\{p_i\} \times \mathbf{CP}^1) \cup \bigcup_{j=1}^{2}(\mathbf{CP}^1 \times \{q_j\}) \subset \mathbf{CP}^1 \times \mathbf{CP}^1.$$

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Let $D_{2,1}$ be the double branched cover of $\mathbf{CP}^1 \times \mathbf{CP}^1$ branched along $B_{2,1}$. Then $E(1)$ is the desingularization of $D_{2,1}$

$$p : E(1) \to D_{2,1} \to \mathbf{CP}^1 \times \mathbf{CP}^1.$$

Let $pr_1 : \mathbf{CP}^1 \times \mathbf{CP}^1 \to \mathbf{CP}^1$ denote the projection onto the first factor and define $\pi = pr_1 \circ p$. Then $\pi : E(1) \to \mathbf{CP}^1$ is a fibration with generic fiber $\mathbf{CP}^1$ since the generic fiber of pr_1 meets the branch locus $B_{2,1}$ at two points. On the other hand, note that the composition $pr_2 \circ p$ is the standard elliptic fibration, where pr_2 is the projection to the second factor. We will denote the generic torus fiber of $pr_2 \circ p$ by F. For more details on these two fibrations, we refer to [GS] (§7.3).

Next we perform a knot surgery on $E(1)$ as in [FS2]. Let K be the trefoil knot in S^3 and m a meridional circle to K. Perform the 0-framed surgery on K and call the resulting 3-manifold $S^3(K)$. In $S^3(K) \times S^1$ we have the smoothly embedded torus $T = m \times S^1$ of self-intersection 0. Let $E(1)_K$ denote the fiber sum

$$E(1)\#_{F=T}(S^3(K) \times S^1) = [E(1) - (F \times D^2)] \cup_\psi [(S^3(K) \times S^1) - (T \times D^2)],$$

where the two pieces are glued together so as to preserve the homology class $\alpha = [\{pt\} \times \partial D^2]$. Fintushel and Stern showed that $E(1)_K$ is in fact a symplectic 4-manifold homotopy equivalent to $E(1)$.

Now let S denote the generic rational fiber of π. Note that $[S] \cdot [F] = 2$, i.e. S geometrically intersects the elliptic fiber F at two points. Recall that the Seifert surface Σ^0 of the trefoil knot K is a punctured torus. Since the gluing map ψ sends the homology class α into the homology class $[K \times \{pt\}] = [\partial\Sigma^0 \times \{pt\}]$ in $(S^3(K) \times S^1) - (T \times D^2)$, we can glue together two copies of Σ^0 and the twice-punctured rational fiber $S - (F \times D^2)$, and get a smooth genus 2 surface Σ_2 in $E(1)_K$. In this way π induces a genus 2 fibration $E(1)_K \to \mathbf{CP}^1$ with generic fiber Σ_2, which we continue to denote by π. Note that the fibration $\pi : E(1)_K \to \mathbf{CP}^1$ has no multiple fibers. After a small perturbation we can assume that π is a Lefschetz fibration (cf. [GS], Chapter 8). We can further assume that the vanishing cycle on the fiber Σ_2 is homologically nontrivial.

There was nothing special about the trefoil knot in the above construction and we can easily generalize our result to the following

Lemma 2.2 (See [F]). *Suppose L is a fibered knot in S^3 of genus $g(L)$. Then there is a genus $2g(L)$ Lefschetz fibration $\pi : E(1)_L \to \mathbf{CP}^1$.* □

Our second building block W will be the fiber sum of two copies of the Kodaira-Thurston manifold. This is the manifold Q_2 in [G] (p. 570). Recall that W fibers over a genus 2 surface and the fibration admits a section Σ of self-intersection 0. Note that W has a symplectic structure and Σ is a symplectic submanifold. Also recall from [Sz] (Lemma 2.1, p. 413) that there is an element $g \in \pi_1(\Sigma)$ such that the monodromy relation $g^{-1}ag = ab$ holds inside $\pi_1(W)$. Here, (a, b) denotes a set of generators for the fundamental group of the fiber in W.

Now we have all the ingredients needed for our construction. Our manifold X_{11} will be the symplectic sum of W and $E(1)_K$ along Σ and Σ_2. We identify the tubular neighborhoods N_1 of Σ and N_2 of Σ_2 via a diffeomorphism $\varphi : N_1 - \Sigma \to N_2 - \Sigma_2$, which preserves the orientations on the normal disks. We choose the gluing map φ in such a way that φ maps the generator g to the vanishing cycle on the fiber Σ_2

in $E(1)_K$. Let

$$X_{11} = W\#_\varphi E(1)_K = (W - \Sigma) \cup_\varphi (E(1)_K - \Sigma_2),$$

where we use φ to identify $N_1 - \Sigma$ and $N_2 - \Sigma_2$. For more on the details of the symplectic sum operation, we refer to [G].

Lemma 2.3. (i) *X_{11} is a smooth, closed, oriented, symplectic 4-manifold.*
(ii) *X_{11} is simply-connected.*
(iii) *X_{11} is homeomorphic to $3\mathbf{CP}^2\#11\overline{\mathbf{CP}}^2$.*
(iv) *X_{11} is not diffeomorphic to $3\mathbf{CP}^2\#11\overline{\mathbf{CP}}^2$.*

Proof. Part (i) is immediate. Since $E(1)_K$ is simply-connected, an easy application of van Kampen's theorem says that $\pi_1(E(1)_K - \Sigma_2)/\langle\mu\rangle = 1$, where μ denotes the meridian of Σ_2. Now note that $\mu = [a, b]$ in $\pi_1(W\#_\varphi E(1)_K)$. But in the group $\pi_1(X_{11})$, we have $g = 1$ which implies that $b = 1$, which in turn implies that $\mu = 1$. We conclude that the homomorphism, $\pi_1(E(1)_K - \Sigma_2) \to \pi_1(X_{11})$, induced by the inclusion map is the zero homomorphism.

It can be shown (cf. [G], p. 571) that $\pi_1(W - \Sigma)/\langle\pi_1(\Sigma^{\parallel})\rangle = 1$, where $\Sigma^{\parallel}$ is a parallel copy of Σ in $(W - \Sigma)$. Since Σ gets identified with Σ_2 in $E(1)_K$ and the composition of inclusions, $\Sigma^{\parallel} \hookrightarrow (E(1)_K - \Sigma_2) \hookrightarrow X_{11}$, induces the zero map on the fundamental groups, an easy application of van Kampen's theorem gives (ii).

Various topological invariants behave nicely under the symplectic sum operation (cf. [G], p. 535):

$$sign(X_{11}) = sign(W) + sign(E(1)_K) = sign(E(1)_K) = -8,$$

$$e(X_{11}) = e(W) + e(E(1)_K) - 2e(\Sigma) = e(E(1)_K) + 4 = 16.$$

It follows that $b_2^+(X_{11}) = 3$ and $b_2^-(X_{11}) = 11$. Hence X_{11} has an odd intersection form $3\langle 1\rangle \oplus 11\langle -1\rangle$. Since X_{11} is smooth, (iii) follows from Freedman's famous classification theorem (cf. [FQ]). Part (iv) follows from Proposition 2.1 since $3\mathbf{CP}^2\#11\overline{\mathbf{CP}}^2$ admits a metric of positive scalar curvature (see e.g. [Sa]). □

Now we proceed with the construction of X_{13}. This time around, we keep the homotopy rational elliptic surface summand $E(1)_K$ in the construction of X_{11}, but replace the W summand with $T^4\#2\overline{\mathbf{CP}}^2$. Choose four distinct points $z_i \in S^1$, $i = 1, \dots, 4$. Let us define smoothly embedded 2-tori in the 4-torus $T_{12} \subset T^4$, $T_{34} \subset T^4$ by

$$T_{12} = S^1 \times S^1 \times \{z_1\} \times \{z_2\}, \;\; T_{34} = \{z_3\} \times \{z_4\} \times S^1 \times S^1.$$

We choose a symplectic form on T^4 for which T_{12}, T_{34} are symplectic submanifolds. Note that $[T_{12}] \cdot [T_{34}] = 1$, and $[T_{12}]^2 = [T_{34}]^2 = 0$. By symplectically resolving the intersection of T_{12} and T_{34}, and then blowing up twice to reduce the self-intersection, we obtain a symplectic genus 2 surface $\tilde{\Sigma} \hookrightarrow T^4\#2\overline{\mathbf{CP}}^2$ with self-intersection 0. We denote $T^4\#2\overline{\mathbf{CP}}^2$ by V. Note that $\pi_1(V) \cong \pi_1(T^4) \cong \mathbf{Z}^4$, $e(V) = 2$, and $sign(V) = -2$.

As before, we symplectically sum V and $E(1)_K$ along $\tilde{\Sigma}$ and Σ_2, and define

$$X_{13} = V\#_\varphi E(1)_K = (V - \tilde{\Sigma}) \cup_\varphi (E(1)_K - \Sigma_2),$$

where φ is some suitably chosen diffeomorphism as before.

Lemma 2.4. (i) *X_{13} is a smooth, closed, oriented, symplectic 4-manifold.*
(ii) *X_{13} is simply-connected.*
(iii) *X_{13} is homeomorphic to* $3\mathbf{CP}^2\#13\overline{\mathbf{CP}}^2$.
(iv) *X_{13} is not diffeomorphic to* $3\mathbf{CP}^2\#13\overline{\mathbf{CP}}^2$.

Proof. Part (i) is immediate. The proofs of the other parts mirror the proofs of the corresponding statements for X_{11}. For (ii), we note that if $\tilde{\Sigma}^{\|}$ is a parallel copy of $\tilde{\Sigma}$ in $(V-\tilde{\Sigma})$, then the inclusion induces a surjection $\pi_1(\tilde{\Sigma}^{\|})\to\pi_1(V-\tilde{\Sigma})$. In particular, the meridian of $\tilde{\Sigma}$ is killed by an embedded disk coming from the exceptional divisor of a blow-up, and hence we can proceed as before. For (iii), we calculate that $e(X_{13})=18$, $sign(X_{13})=-10$, and then invoke Freedman's theorem. Since $3\mathbf{CP}^2\#13\overline{\mathbf{CP}}^2$ admits a metric of positive scalar curvature, Proposition 2.1 implies (iv). □

We have thus proved Theorem 1.1 for the odd cases $n=11,13$ except for the irreducibility condition which will be proved in the next section.

3. Irreducibility and SW-invariants

Recall that a smooth closed simply-connected 4-manifold X is called *irreducible* if each connected sum decomposition of X as $X=Y\#Z$ satisfies that either Y or Z is a homotopy S^4. Irreducibility of X_n will follow easily from the computation of SW-invariants of X_n using the product formula of [MST]. In order to use the product formula we must first compute the SW-invariants of each summand. We keep the conventions in Section 3 of [P] regarding the choice involved in the definition of SW-invariant for the $b_2^+=1$ case. A lot of times, we abuse notation and use the same capital letter to denote a surface, its homology class, or the Poincaré dual of its homology class.

Lemma 3.1. *Let T denote the Poincaré dual of the homology class of the torus fiber in $E(1)$. Then the only SW-basic classes of $E(1)_K$ are $\pm T$.*

Proof. We refer the reader to the last section of [FS2], where a more general statement is proved for the case when K is an arbitrary twist knot. Note that the trefoil is (-1)-twist knot. □

Lemma 3.2. *Let F^* denote the Poincaré dual of the homology class of the fiber in W. Then the only possible SW-basic classes of W are $0,\pm 2F^*$.*

Proof. This is an easy application of the generalized adjunction inequality for the SW-basic classes (cf. [OS]), using the fact that $H_2(W;\mathbf{Z})\cong\mathbf{Z}^6$ (cf. [Sz], p. 413) has generators consisting of the section Σ and five tori (one of which is the fiber F). □

Lemma 3.3. *The only SW-basic classes of V are $\pm D_1\pm D_2$, where D_i are Poincaré-dual to the exceptional divisors of the blow-ups.*

Proof. By the generalized adjunction inequality, 0 is the only SW-basic class of T^4. Now apply the blow-up formula for SW-invariants (cf. [FS1]). □

Theorem 3.4. *Let $K_{X_{11}}$ denote the canonical class of the symplectic structure on X_{11}. Then $SW_{X_{11}}(\pm K_{X_{11}})=\pm 1$, and $SW_{X_{11}}(L)=0$ if $L\neq\pm K_{X_{11}}$.*

Proof. The first statement is proved in [T]. It remains to prove that there are no other basic classes. Choose a horizontal sphere in $B_{2,1}$, say $\mathbf{CP}^1 \times \{q_1\}$, and recall that there is a curve $R \subset E(1)$ of self-intersection $R^2 = -2$ such that $p(R) = \mathbf{CP}^1 \times \{q_1\}$ (cf. [GS], §7.3). Note that $R \cdot \Sigma_2 = 1$ in $E(1)_K$. If F denotes the fiber of W, then we have $F \cdot \Sigma = 1$. Since the identification $\Sigma = \Sigma_2$ is made in X_{11}, we can cut out a small disk from both F and R, centered at their intersection with Σ and Σ_2 respectively, and then glue together the resulting open surfaces F^0 and R^0 along a meridional tube of $\Sigma = \Sigma_2$ to get a new smooth surface Γ. Inside X_{11}, we have $\Gamma \cdot \Gamma = -2$, and $\Gamma \cdot \Sigma = 1$. Now $b_2(X_{11}) = 14$ and we easily get the following orthogonal decomposition of $H_2(X_{11})$ with respect to the intersection pairing

$$H_2(X_{11}; \mathbf{Z}) \cong \mathcal{T} \oplus \langle \Sigma, \Gamma \rangle \oplus \mathcal{N}.$$

Here, $\mathcal{T} \cong \mathbf{Z}^4$ comes from $\langle \Sigma, F \rangle^\perp \subset H_2(W)$, and has generators consisting of four tori of self-intersection 0. $\mathcal{N} \cong \mathbf{Z}^8$ is negative definite and comes from $\langle \Sigma_2, R \rangle^\perp \subset H_2(E(1)_K)$. Note that the generators of $\mathcal{T}$ and $\mathcal{N}$ lie away from the gluing area of the symplectic sum operation $\#_\varphi$.

Now suppose $SW_{X_{11}}(L) \neq 0$. Then by repeated applications of the generalized adjunction inequality, L is orthogonal to $\mathcal{T}$, i.e.

$$L = a\Sigma + b\Gamma + \nu,$$

where $\nu \in \mathcal{N}$ and $b = 0, \pm 2$. But X_{11} is symplectic, hence of SW-simple type. This implies that $L^2 = 2e(X_{11}) + 3sign(X_{11}) = 8$. Since $2ab \geq L^2$, we must have $b \neq 0$. Now

$$\langle \pm L, \Sigma \rangle = \pm b = 2 = 2g(\Sigma) - 2$$

so we can apply the product formula in [MST]. From the computations in two previous lemmas applied to the product formula, we easily see that L must now be orthogonal to $\mathcal{N}$, i.e. $L = a\Sigma \pm 2\Gamma$. From $L^2 = 2ab - 2b^2 = 8$, we conclude that $L = \pm(4\Sigma + 2\Gamma)$. By the pigeonhole principle,

$$L = \pm K_{X_{11}} = \pm(4\Sigma + 2\Gamma).$$

□

Theorem 3.5. *Let $K_{X_{13}}$ denote the canonical class of the symplectic structure on X_{13}. Then $SW_{X_{13}}(\pm K_{X_{13}}) = \pm 1$, and $SW_{X_{13}}(L) = 0$ if $L \neq \pm K_{X_{13}}$.*

Proof. This proof is completely analogous to the argument made for X_{11}. We use the previous cut-and-paste method to make sense out of the expressions $[T_{12}\#R]$, $[T_{34}\#R]$, $[(D_1 + D_2)\#T]$, etc. (Here, T is the elliptic fiber of $E(1)$.) For example, $[T_{12}\#R]$ and $[T_{34}\#R]$ are represented by smooth tori of square (-2) inside X_{13}. If L is a basic class of X_{13}, then $L^2 = 6$. From the generalized adjunction inequality [OS] and the product formula [MST], it easily follows that

$$L = \pm(3[\tilde{\Sigma}] + [T_{12}\#R] + [T_{34}\#R]) = \pm K_{X_{13}}.$$

□

Corollary 3.6. *X_{11} and X_{13} are irreducible.*

Proof. Since X_n has nontrivial Seiberg-Witten invariants, it follows that every connected sum decomposition of X_n as $X_n = Y\#Z$ satisfies that one of the pieces, say Z, is a homotopy $k\overline{\mathbf{CP}}^2$ with some $k \geq 0$. If $k > 0$, then the blow-up formula for SW-invariants implies the existence of SW-basic classes L, L' of X_n with $(L - L')^2 = -4$. But this is easily seen to be impossible. □

Remark 3.7. The connected sum theorem for the Seiberg-Witten invariant (cf. [Sa]) can be invoked to say that the SW-invariant of $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ is 0. Combined with Theorems 3.4 and 3.5, this gives an alternative proof that X_n is not diffeomorphic to $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$.

4. Appendix: Nonexistence of certain holomorphic curves

In the preliminary draft of this paper, the author constructed X_n on the assumption that the Barlow surface (cf. [B]) contains a genus 2 holomorphic curve of self-intersection 1. It turns out that this assumption is false and one has the following

Proposition 4.1. *There is no holomorphic genus* 2 *curve of square* 1 *inside the Barlow surface* B*.*

Proof. Recall that B is homeomorphic to $\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$ with $K^2 = 1$ and $q = p_g = 0$. Choose an orthogonal basis $\{H, E_1, \ldots, E_8\}$ of $H_2(B;\mathbf{Z})$ such that $H^2 = 1$ and $E_i^2 = -1$. After a change of basis we can assume that the canonical class K is a *reduced class* (cf. [LL], p. 576). Since $K^2 = 1$, there are only two possibilities, namely $K = H$ or $K = 3H - \sum_{i=1}^8 E_i$. Now suppose $C = aH - \sum_{i=1}^8 b_i E_i$ is a genus 2 holomorphic curve of square $C^2 = 1$. By the adjunction formula, we must have $K \cdot C = 1$. If $K = H$, then we immediately see that $C = H$. If $K = 3H - \sum_{i=1}^8 E_i$, then we must have

$$\begin{cases} 3a - \sum_{i=1}^8 b_i = 1, \\ a^2 - \sum_{i=1}^8 b_i^2 = 1. \end{cases}$$

By the Cauchy-Schwartz inequality, $(3a-1)^2 \leq 8(a^2 - 1)$, which implies $a = 3$. It easily follows that $C = K$ in this case as well. But $p_g = \dim_{\mathbf{C}} \Gamma(B, \Lambda^{2,0}T^*B) = 0$, hence there cannot be a holomorphic curve representing $K = c_1(\Lambda^{2,0}T^*B)$. □

Remark 4.2. In fact the above proposition continues to hold for an arbitrary simply-connected numerical Godeaux surface.

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References

[B] R. Barlow: A simply connected surface of general type with $p_g = 0$, *Invent. math.* **79** (1985), 293-301. MR **87a:**14033

[F] R. Fintushel: Lecture at the Topology and Geometry Conference, University of Århus, 1998.

[FS1] R. Fintushel and R.J. Stern: Immersed Spheres in 4-Manifolds and The Immersed Thom Conjecture, *Turkish J. of Math.* **19** (1995), 145-157. MR **96j:**57036

[FS2] R. Fintushel and R.J. Stern: Knots, Links and 4-manifolds, to appear in *Invent. math.*

[FQ] M.H. Freedman and F. Quinn: Topology of 4-Manifolds, Princeton University Press, 1990. MR **94b:**57021

[G] R.E. Gompf: A new construction of symplectic manifolds, *Annals of Math.* **142** (1995), 527-595. MR **96j:**57025

[GS] R.E. Gompf and A.I. Stipsicz: An Introduction to 4-Manifolds and Kirby Calculus, *preprint.*

[LL] B.H. Li and T.J. Li: Minimal genus smooth embeddings in $S^2 \times S^2$ and $\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ with $n \leq 8$, *Topology* **37** (1998), 575-594. MR **99b:**57059

[MST] J.W. Morgan, Z. Szabó and C.H. Taubes: A Product Formula for the Seiberg-Witten Invariants and the Generalized Thom Conjecture, *J. Diff. Geom.* **44** (1996), 706-788. MR **97m:**57052

[OS] P. Ozsváth and Z. Szabó: The symplectic Thom conjecture, *preprint.*

[P] B.D. Park: Exotic smooth structures on $3\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$, this issue.

[Sa] D. Salamon: Spin Geometry and Seiberg-Witten Invariants, *preprint.*

[Sz] Z. Szabó: Irreducible four-manifolds with small Euler characteristics, *Topology* **35** (1996), 411-426. MR **97c:**57021

[T] C.H. Taubes: The Seiberg-Witten invariants and symplectic forms, *Math. Res. Letters* **1** (1994), 809-822. MR **95j:**57039

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