INCOMPRESSIBLE SURFACES IN HANDLEBODIES
AND CLOSED 3-MANIFOLDS OF HEEGAARD GENUS 2

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Abstract. In this paper, we shall prove that for any integer $n > 0$, 1) a handlebody of genus 2 contains a separating incompressible surface of genus $n$, 2) there exists a closed 3-manifold of Heegaard genus 2 which contains a separating incompressible surface of genus $n$.

1. Introduction

Let $M$ be a 3-manifold, and let $F$ be a properly embedded surface in $M$. $F$ is said to be compressible if either $F$ is a 2-sphere and $F$ bounds a 3-cell in $M$, or there exists a disk $D \subset M$ such that $D \cap F = \partial D$, and $\partial D$ is nontrivial on $F$. Otherwise $F$ is said to be incompressible.

W. Jaco (see [3]) has proved that a handlebody of genus 2 contains a nonseparating incompressible surface of arbitrarily high genus, and asked the following question.

Question A. Does a handlebody of genus 2 contain a separating incompressible surface of arbitrarily high genus?

In the second section, we shall give an affirmative answer to this question. The main result is the following.

Theorem 2.7. A handlebody of genus 2 contains a separating incompressible surface $S$ of arbitrarily high genus such that $|\partial S| = 1$.

If $M$ is a closed 3-manifold of Heegaard genus 1, then $M$ is homeomorphic to either a lens space or $S^2 \times S^1$. In the third section, we shall prove that the Heegaard genus of a closed 3-manifold $M$ does not limit the genus of an incompressible surface in $M$. The main result is the following.

Theorem 3.10. For any integer $n > 0$, there exists a closed 3-manifold of Heegaard genus 2 which contains a separating incompressible surface of genus $n$.

If $X$ is a manifold, we shall denote by $\partial X$ the boundary of $X$, and by $|\partial X|$ the number of components of $\partial X$. If $F(x_1, \ldots, x_n)$ is a free group, and $y$ is an element...
in $F$, we shall denote by $L(y)$ the minimal length of $y$ with respect to the basis $x_1, \ldots, x_n$.

Some examples answering Jaco’s question have also been given by Hugh Howards in “Generating disjoint incompressible surfaces”, preprint, 1998.

2. INCOMPRESSIBLE SURFACES IN HANDLEBODIES

Let $H_2$ be a handlebody of genus 2, and let $(D_1, D_2)$ be a set of basis disks of $H_2$. Let $D$ be a separating disk of $H_2$ such that $D_1$ and $D_2$ lie on opposite sides of $D$. Suppose that $u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}$ are $4n$ points on $\partial D$ as in Figure 1.

Suppose that $u_1 v_1, \ldots, u_{2n} v_{2n}$ are $2n$ arcs on $\partial H_2$ such that
1) $u_1 v_1$ is as in Figure 2,
2) $u_2 v_{2i-1}$ and $u_{2i-1} v_{2i}$ are as in Figure 3,
3) $u_{2i+1} v_{2i}$ and $u_{2i} v_{2i+1}$ are as in Figure 3 and
4) $u_k v_k$ is the union of $u_k v_{k-1}, v_{k-1} u_{k-1}$ and $u_{k-1} v_k$.

Then $u_i v_i \subset u_{i+1} v_{i+1}$.

Suppose that $N_k = u_k v_k \times B_k$, where $B_k$ is a half disk in $H_2$ (as in Figure 4) such that
1) $\{u_i\} \times B_k \cup \{v_i\} \times B_k \subset D(i \leq k)$, and
2) if $x \in u_k v_k$, then $\{x\} \times B_i \subset \text{int}(\{x\} \times B_k)(i > k)$.

Let $C_0 = D - \bigcup_{i=1}^{k-1}(\{u_i\} \times B_i) - \bigcup_{i=1}^{k-1}(\{v_i\} \times B_i)$, and $D_0 = \tilde{C}_0$. Let $C_i = \partial(u_i v_i \times B_i) - \partial H_2 - \{u_i\} \times B_i - \{v_i\} \times B_i$, and $D_i = \tilde{C}_i$, where $1 \leq i \leq k$. Let $S_k = \bigcup_{i=0}^{k} D_i$. Then $S_k$ is a properly embedded surface in $H_2$ ($1 \leq k \leq 2n$). Let
$A_i = D_0 \cup D_i \ (i > 0)$. Then $A_i$ is an annulus. Hence $S_k$ is a union of $k$ annuli $A_1, \ldots, A_k$ along $D_0$.

**Lemma 2.1.** $|\partial S_{2k-1}| = 2$ and $|\partial S_{2k}| = 1 \ (1 \leq k \leq n)$.

**Proof.** It is clear that $|\partial S_1| = 2$. Since $u_2$ and $v_2$ lie in the two distinct components of $\partial S_1$, by construction, $|\partial S_2| = 1$. We can prove that $|\partial S_{2k-1}| = 2$ and $|\partial S_{2k}| = 1$ by induction, for $1 \leq k \leq n$. \hfill $\square$

**Lemma 2.2.** The genus of $S_{2n}$ is $n$.

**Proof.** By construction, $S_{2n}$ is a union of $2n$ annuli $A_1, \ldots, A_{2n}$ along $D_0$. Hence $\pi_1(S_{2n}) = F(x_1, \ldots, x_{2n})$, where $x_i$ is represented by the core of $A_i$. Since $|\partial S_{2n}| = 1$, the genus of $S_{2n}$ is $n$. \hfill $\square$

Let $i : S_{2n} \longrightarrow H_2$ be the inclusion map, and $i_* : \pi_1(S_{2n}) \longrightarrow \pi_1(H_2)$ be the map induced by $i$. Suppose that $a_1$ and $a_2$ are the two generators of $\pi_1(H_2)$ shown in Figure 4. Then by construction we have

$i_*(x_1) = a_1^4,$

$i_*(x_2) = a_2^{-1} a_1^{-4} a_2^{-1},$

$i_*(x_{2i}) = a_2^{-1} (a_2 a_1)^{1-i} a_1^{-4} (a_1 a_2)^{1-i} a_2^{-1},$

$i_*(x_{2i+1}) = (a_1 a_2)^i a_1^4 (a_2 a_1)^i,$

$i_*(x_{2n}) = a_2^{-1} (a_2 a_1)^{1-n} a_1^{-4} (a_1 a_2)^{1-n} a_2^{-1}.$
Let $y$ be a nontrivial element of $\pi_1(S_{2n})$. Then $y = \prod_{i=1}^{m} b_i$, where $b_i = \prod_{j=1}^{p_{ij}} b_{ij}$, $b_{ij} \in \{x_1, x_2, \ldots, x_{2n}\}$ and $m_i \geq 1$, such that

1) $p_{ij} \neq 0$.
2) for each $i, 1 \leq i \leq m$, either all the $b_{ij}$'s ($1 \leq j \leq m_i$) belong to $\{x_1, x_3, \ldots, x_{2n-1}\}$ or they all belong to $\{x_2, x_4, \ldots, x_{2n}\}$.
3) the $b_{ij}$'s belong to $\{x_1, x_3, \ldots, x_{2n-1}\}$ if and only if the $b_{i+1,j}$'s belong to $\{x_2, x_4, \ldots, x_{2n}\}$, and
4) $b_{ij} \neq b_{i+1,j+1}$.

Lemma 2.3. If $b_{ij} \in \{x_1, \ldots, x_{2n-1}\}$ for $j \in \{1, \ldots, m_i\}$, then $L(i_*(b_i)) > 1$. Also, the first letter of $i_*(b_i)$ is $a_1$ or $a_1^{-1}$, and the last letter of $i_*(b_i)$ is $a_2$ or $a_2^{-1}$.

Proof. Suppose that $b_{ij} = x_{2l_j+1}$, where $l_j \in \{0, 1, \ldots, n-1\}$. Then

$$i_*(b_{ij}) = (a_1 a_2)^{l_j} a_1^k (a_2 a_1)^{l_j} \cdots (a_1 a_2)^{l_j} a_1^k (a_2 a_1)^{l_j},$$ or

$$i_*(b_{ij}) = (a_2 a_1)^{-l_j} a_1^{-4}(a_1 a_2)^{-l_j} \cdots (a_2 a_1)^{-l_j} a_1^{-4}(a_1 a_2)^{-l_j}.$$ If $p_{ij} > 0$, and $p_{ij+1} > 0$, then

$$i_*(b_{ij+1}) = (a_1 a_2)^{l_j} a_1^{k} (a_2 a_1)^{l_j} (a_1 a_2)^{l_j+1} a_1^k (a_2 a_1)^{l_j+1}.$$ If $p_{ij} > 0$, and $p_{ij+1} < 0$, then

$$i_*(b_{ij+1}) = (a_1 a_2)^{l_j} a_1^{k} (a_2 a_1)^{l_j} (a_1 a_2)^{-l_j+1} a_1^{-4}(a_1 a_2)^{-l_j+1}.$$ Since $b_{ij} \neq b_{ij+1}$, $l_j \neq l_{j+1}$. It is easy to see that the first and last letters of $i_*(b_i)$ are $a_1$ or $a_1^{-1}$, and $L(i_*(b_i)) \geq 2$.

Lemma 2.4. If $b_{ij} \in \{x_2, \ldots, x_{2n}\}$, then $L(i_*(b_i)) > 1$. Also, the first letter of $i_*(b_i)$ is $a_2$ or $a_2^{-1}$, and the last letter of $i_*(b_i)$ is $a_2$ or $a_2^{-1}$.

Proof. The proof of Lemma 2.4 is similar to the proof of Lemma 2.3.

Lemma 2.5. $S_{2n}$ is incompressible in $H_2$.

Proof. Suppose that $y$ is a nontrivial element of $\pi_1(S_{2n})$. Then $y = \prod_{i=1}^{m} b_i$, where $b_i$ satisfies the above conditions. By Lemma 2.3 and Lemma 2.4, $L(i_*(y)) = \sum_{i=1}^{m} L(i_*(b_i))$. Hence $L(i_*(y)) > 1$, and $S_{2n}$ is incompressible in $H_2$.

Lemma 2.6. $S_{2n}$ is separating in $H_2$.

Proof. By construction, $[\partial S_{2n}] = 0$ in $H_1(\partial H_2)$. Hence $\partial S_{2n}$ is separating on $\partial H_2$, and $S_{2n}$ is separating in $H_2$.

Theorem 2.7. A handlebody of genus 2 contains a separating incompressible surface $S$ of arbitrarily high genus such that $|\partial S| = 1$.

Proof. Let $n$ be a positive integer, then $S_{2n}$ is a separating incompressible surface in $H_2$ by the above argument.

Remark 1. In fact a handlebody of genus 2 also contains a separating incompressible surface $S$ of arbitrarily high genus such that $|\partial S| = 2$. For example, $S_{2n-1}$ is a separating incompressible surface of genus $n - 1$ in $H_2$.

Remark 2. For any positive integer $n$, there exist infinitely many separating incompressible surfaces of genus $n$ in a handlebody of genus 2. For example, let $n \geq 3$. Then by the proof of Theorem 2.7 we can obtain another separating incompressible surface of genus $n$ in $H_2$ by the same method as in the above construction.
Corollary 2.8. For any integer $n > 1$, a handlebody of genus $n$ contains a separating incompressible surface $S$ of arbitrarily high genus.

3. Incompressible surfaces in closed 3-manifolds of Heegaard genus 2

Let $M$ be a compact 3-manifold with boundary. If $c_1, \ldots, c_n$ are disjoint simple closed curves on $\partial M$, we denote by $\tau(M, \bigcup_{i=1}^{n} c_i)$ the manifold obtained by attaching 2-handles to $M$ along disjoint regular neighborhoods of $c_1, \ldots, c_n$, and $M[c_1] \cdots [c_n]$ the manifold obtained by capping off possible 2-sphere components of $\partial \tau(M, \bigcup_{i=1}^{n} c_i)$. If $c$ is a nontrivial simple closed curve on a toral component of $\partial M$, we denote by $M(c)$ the manifold $M[c]$. Now if $F(\neq S^2)$ is a separating incompressible closed surface in $\tau(M, \bigcup_{i=1}^{n} c_i)$, then $F$ is also a separating incompressible surface in $M[c_1] \cdots [c_n]$.

If $S$ is a properly embedded surface in $M$, we denote by $\hat{S}$ the surface obtained by capping off the boundary components of $S$ with disks in $\tau(M, \partial S)$.

Lemma 3.1. $H_2$ contains no closed incompressible surface.

Proof. Let $D$ be a properly embedded disk in $H_2$ such that $\partial D$ is nontrivial on $\partial H_2$. If $F$ is a closed incompressible surface, then $F$ may be isotoped to be disjoint from $D$. Hence a 3-cell contains a closed incompressible surface, a contradiction.

Lemma 3.2. Let $M$ be a 3-manifold, and let $J$ be a simple closed curve on $\partial M$ such that $\partial M - J$ is incompressible. If $M$ has compressible boundary, then $\tau(M, J)$ is a $\partial$-irreducible manifold.

Proof. See [4, Theorem 2].

Let $H_2$ be a handlebody of genus 2, and let $S$ be a separating incompressible surface of genus $n > 0$ in $H_2$ such that $|\partial S| = 1$.

Lemma 3.3. $\tau(H_2, \partial S)$ is a $\partial$-irreducible 3-manifold, and $\hat{S}$ is a separating closed incompressible surface of genus $n$ in $\tau(H_2, \partial S)$.

Proof. Suppose that $S$ separates $H_2$ into $H, H'$ and $\partial S$ separates $\partial H_2$ into $T, T'$, such that $\partial H = S \cup T$ and $\partial H' = S \cup T'$.

Claim 1. $\partial H(\partial H')$ is compressible in $H(H')$.

Proof. If $\partial H$ is incompressible in $H$, then $\partial H$ is incompressible in $H_2$, contradicting Lemma 3.1.

Claim 2. $T(T')$ is incompressible in $H(H')$.

Proof. If $T$ is compressible in $H$, then there exists a nontrivial simple closed curve $c$ on $T$ such that $c$ bounds a disk $D$ in $H$. Since the genus of $\partial H_2$ is 2, $T$ is a once punctured torus whose boundary is isotopic to $\partial S$. Hence $\partial S$ bounds a disk in $H_2$, a contradiction.

Since $\partial H(\partial H')$ is compressible in $H(H')$, and $S$ and $T(T')$ are incompressible in $H(H')$, it follows that $\tau(H, \partial S)$ and $\tau(H', \partial S)$ are $\partial$-irreducible 3-manifolds by Lemma 3.2.

Since $\tau(H_2, \partial S) = \tau(H, \partial S) \cup \tau(H', \partial S)$, $\hat{S}$ is a separating incompressible closed surface in $\tau(H_2, \partial S)$. 

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Since the genus of \( \partial H_2 \) is 2, \( \partial \tau (H_2, \partial S) \) consists of two tori, \( T_1 \) and \( T_2 \), say.

Let \( M \) be a 3-manifold with one component \( T \) of \( \partial M \) a torus. If \( r_1 \) and \( r_2 \) are two slopes on \( T \), we shall denote by \( \Delta (r_1, r_2) \) the minimal geometric intersection number among all the curves representing the slopes.

**Lemma 3.4.** Let \( M \) be a \( \partial \)-irreducible 3-manifold with one component \( T \) of \( \partial M \) a torus, and let \( F \) be a closed incompressible surface in \( M \) which is not parallel to \( T \). If \( r_1 \) and \( r_2 \) are two slopes on \( T \) such that \( F \) is compressible in \( M(r_1) \) and \( M(r_2) \), then either

1) \( \Delta (r_1, r_2) \leq 1 \), or
2) there exists a slope \( r \) on \( T \) such that \( \Delta (r, r_1) \leq 1 \) and \( \Delta (r, r_2) \leq 1 \).

**Proof.** See [6, Theorem 1].

**Corollary 3.5.** Let \( M \) be a \( \partial \)-irreducible 3-manifold with one component \( T \) of \( \partial M \) a torus. If \( F \) is a closed incompressible surface in \( M \) which is not parallel to \( T \), then there exists a nontrivial simple closed curve \( c \) on \( T \) such that \( F \) is incompressible in \( M(c) \).

**Lemma 3.6.** There exist two nonseparating simple closed curves \( c_1 \) and \( c_2 \) on \( \partial H_2 \) such that \( c_1 \) is disjoint from \( \partial S \), and \( \tilde{S} \) is a separating incompressible surface in \( H_2[\partial S][c_1][c_2] \).

**Proof.** Suppose that \( \tilde{S} \) separates \( \tau (H_2, \partial S) \) into \( M_1 \) and \( M_2 \) such that \( T_1 \subset M_1 \) and \( T_2 \subset M_2 \). By the proof of Lemma 3.1, \( M_i \) is \( \partial \)-irreducible. Since \( \tilde{S} \) is not parallel to \( T_i \) in \( M_i \), by Corollary 3.5, there exists a simple closed curve \( c_i \) (\( 1 \leq i \leq 2 \)) on \( T_i \) such that \( \tilde{S} \) is incompressible in \( M_i(c_i) \).

By an isotopy, we can suppose that \( c_i \) is disjoint from \( \partial S \). Hence \( \tilde{S} \) is incompressible in \( H_2[\partial S][c_1][c_2] \). It is easy to see that \( \tilde{S} \) is separating in \( H_2[\partial S][c_1][c_2] \). \( \square \)

**Definition 3.7.** Two simple closed curves \( \alpha \) and \( \beta \) on \( \partial M \) are said to be coplanar if some component of \( \partial M - \alpha \cup \beta \) is an annulus or a once punctured annulus.

**Lemma 3.8.** Suppose that \( \alpha \) is a nonseparating curve on \( \partial M \). If a separating curve \( \beta \) on \( \partial M \) is coplanar to \( \alpha \), then \( M[\alpha] = M[\beta][\alpha] \).

**Proof.** See [5, Lemma 5.1]. \( \square \)

**Lemma 3.9.** \( H_2[\partial S][c_1][c_2] = H_2[c_1][c_2] \).

**Proof.** Since \( c_1 \) is coplanar to \( \partial S \) on \( \partial H_2 \), we have \( H_2[\partial S][c_1][c_2] = H_2[c_1][c_2] \). \( \square \)

**Theorem 3.10.** For any integer \( n > 0 \), there exists a closed 3-manifold \( M \) of Heegaard genus 2 which contains a closed separating incompressible surface of genus \( n \).

**Proof.** Let \( H_2 \) be a handlebody of genus 2, and let \( S \) be a separating incompressible surface of genus \( n \) such that \( |\partial S| = 1 \). Then \( H_2[c_1][c_2] \) contains a separating incompressible surface \( \tilde{S} \) of genus \( n \), where \( c_1 \) and \( c_2 \) are disjoint nonseparating simple closed curves on \( \partial H_2 \) as in Lemma 3.5. Obviously the Heegaard genus of \( H_2[c_1][c_2] \) is 2. \( \square \)

**Corollary 3.11.** Suppose that \( m \geq 2 \). Then for any integer \( n > 0 \) there exists a closed 3-manifold of Heegaard genus \( m \) which contains a closed separating incompressible surface of genus \( n \).
In fact there are infinitely many simple closed curves $c$ on $\partial H_2$ such that $\tau(H_2, c)$ contains a closed separating incompressible surface of genus $n$. This is shown by the following example.

**Example.** Let $H_2$ be a handlebody of genus 2. Let $S_{2_m}$ be an incompressible surface of genus $n$ constructed by the same method as the construction of $S_{2_n}$ (as in Section 2), such that $u_1v_1$ intersects $D_1$ in $m$ points. Then $\hat{S}_{2_m}$ is incompressible in $\tau(H_2, \partial S_{2_m})$.

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**References**


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