PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 10, Pages 3091–3097
S 0002-9939(00)05360-0
Article electronically published on May 2, 2000

INCOMPRESSIBLE SURFACES IN HANDLEBODIES AND CLOSED 3-MANIFOLDS OF HEEGAARD GENUS 2

RUIFENG QIU

(Communicated by Ronald A. Fintushel)

ABSTRACT. In this paper, we shall prove that for any integer $n > 0$, 1) a handlebody of genus 2 contains a separating incompressible surface of genus n, 2) there exists a closed 3-manifold of Heegaard genus 2 which contains a separating incompressible surface of genus n.

1. INTRODUCTION

Let M be a 3-manifold, and let F be a properly embedded surface in M. F is said to be compressible if either F is a 2-sphere and F bounds a 3-cell in M, or there exists a disk $D \subset M$ such that $D \cap F = \partial D$, and ∂D is nontrivial on F. Otherwise F is said to be incompressible.

W. Jaco (see [3]) has proved that a handlebody of genus 2 contains a nonseparating incompressible surface of arbitrarily high genus, and asked the following question.

Question A. Does a handlebody of genus 2 contain a separating incompressible surface of arbitrarily high genus?

In the second section, we shall give an affirmative answer to this question. The main result is the following.

Theorem 2.7. *A handlebody of genus 2 contains a separating incompressible surface S of arbitrarily high genus such that $|\partial S| = 1$.*

If M is a closed 3-manifold of Heegaard genus 1, then M is homeomorphic to either a lens space or $S^2 \times S^1$. In the third section, we shall prove that the Heegaard genus of a closed 3-manifold M does not limit the genus of an incompressible surface in M. The main result is the following.

Theorem 3.10. *For any integer $n > 0$, there exists a closed 3-manifold of Heegaard genus 2 which contains a separating incompressible surface of genus n.*

If X is a manifold, we shall denote by ∂X the boundary of X, and by $|\partial X|$ the number of components of ∂X. If $F(x_1, \dots, x_n)$ is a free group, and y is an element

Received by the editors September 20, 1996 and, in revised form, November 6, 1998.
2000 *Mathematics Subject Classification.* Primary 57N10.
Key words and phrases. Heegaard genus, incompressible surface.
This research was supported in part by the National Natural Science Foundation of China.

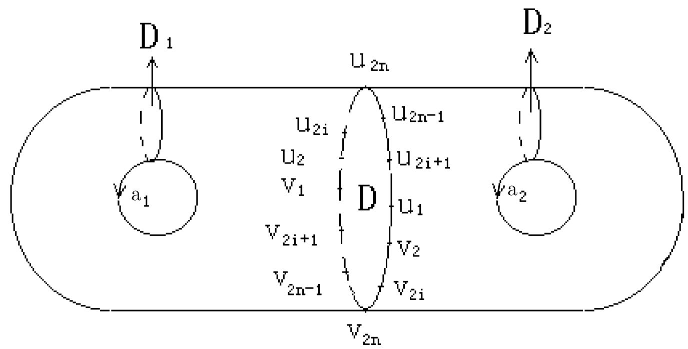

Figure 1.

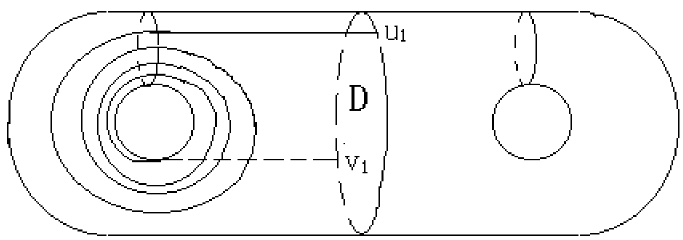

Figure 2.

in F, we shall denote by $L(y)$ the minimal length of y with respect to the basis $x_1, \ldots, x_n$.

Some examples answering Jaco's question have also been given by Hugh Howards in "Generating disjoint incompressible surfaces", preprint, 1998.

2. Incompressible surfaces in handlebodies

Let H_2 be a handlebody of genus 2, and let (D_1, D_2) be a set of basis disks of H_2. Let D be a separating disk of H_2 such that D_1 and D_2 lie on opposite sides of D. Suppose that $u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}$ are $4n$ points on ∂D as in Figure 1.

Suppose that $u_1v_1, \ldots, u_{2n}v_{2n}$ are $2n$ arcs on ∂H_2 such that

1) u_1v_1 is as in Figure 2,

2) $u_{2i}v_{2i-1}$ and $u_{2i-1}v_{2i}$ are as in Figure 3,

3) $u_{2i+1}v_{2i}$ and $u_{2i}v_{2i+1}$ are as in Figure 3, and

4) u_kv_k is the union of u_kv_{k-1}, $v_{k-1}u_{k-1}$ and $u_{k-1}v_k$.

Then $u_iv_i \subset u_{i+1}v_{i+1}$.

Suppose that $N_k = u_kv_k \times B_k$, where B_k is a half disk in H_2 (as in Figure 4) such that

1) $\{u_i\} \times B_k \cup \{v_i\} \times B_k \subset D (i \leq k)$, and

2) if $x \in u_kv_k$, then $\{x\} \times B_i \subset \operatorname{int}(\{x\} \times B_k)(i > k)$.

Let $C_0 = D - \bigcup_{i=1}^{k}(\{u_i\} \times B_i) - \bigcup_{i=1}^{k}(\{v_i\} \times B_i)$, and $D_0 = \bar{C}_0$. Let $C_i = \partial(u_iv_i \times B_i) - \partial H_2 - \{u_i\} \times B_i - \{v_i\} \times B_i$, and $D_i = \bar{C}_i$, where $1 \leq i \leq k$. Let $S_k = \bigcup_{i=0}^{k} D_i$. Then S_k is a properly embedded surface in H_2 $(1 \leq k \leq 2n)$. Let

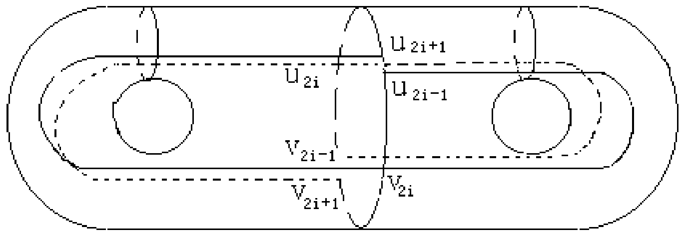

FIGURE 3.

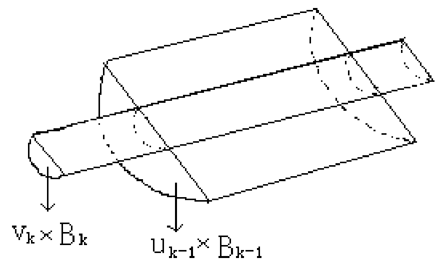

FIGURE 4.

$A_i = D_0 \cup D_i$ $(i > 0)$. Then A_i is an annulus. Hence S_k is a union of k annuli $A_1, \ldots, A_k$ along D_0.

Lemma 2.1. *$|\partial S_{2k-1}| = 2$ and $|\partial S_{2k}| = 1$ $(1 \leq k \leq n)$.*

Proof. It is clear that $|\partial S_1| = 2$. Since u_2 and v_2 lie in the two distinct components of ∂S_1, by construction, $|\partial S_2| = 1$. We can prove that $|\partial S_{2k-1}| = 2$ and $|\partial S_{2k}| = 1$ by induction, for $1 \leq k \leq n$. □

Lemma 2.2. *The genus of S_{2n} is n.*

Proof. By construction, S_{2n} is a union of $2n$ annuli $A_1, \ldots, A_{2n}$ along D_0. Hence $\pi_1(S_{2n}) = F(x_1, \ldots, x_{2n})$, where x_i is represented by the core of A_i. Since $|\partial S_{2n}| = 1$, the genus of S_{2n} is n. □

Let $i : S_{2n} \longrightarrow H_2$ be the inclusion map, and $i_* : \pi_1(S_{2n}) \longrightarrow \pi_1(H_2)$ be the map induced by i. Suppose that a_1 and a_2 are the two generators of $\pi_1(H_2)$ shown in Figure 1. Then by construction we have

$i_*(x_1) = a_1^4$,
$i_*(x_2) = a_2^{-1}a_1^{-4}a_2^{-1}$,
.
.
.
$i_*(x_{2i}) = a_2^{-1}(a_2a_1)^{1-i}a_1^{-4}(a_1a_2)^{1-i}a_2^{-1}$,
$i_*(x_{2i+1}) = (a_1a_2)^i a_1^4 (a_2a_1)^i$,
.
.
.
$i_*(x_{2n}) = a_2^{-1}(a_2a_1)^{1-n}a_1^{-4}(a_1a_2)^{1-n}a_2^{-1}$.

Let y be a nontrivial element of $\pi_1(S_{2n})$. Then $y = \prod_{i=1}^{m} b_i$, where $b_i = \prod_{j=1}^{m_i} b_{ij}^{p_{ij}}, b_{ij} \in \{x_1, x_2, \dots, x_{2n}\}$ and $m_i \geq 1$, such that

1) $p_{ij} \neq 0$,

2) for each $i, 1 \leq i \leq m$, either all the b_{ij}'s $(1 \leq j \leq m_i)$ belong to $\{x_1, x_3, \dots, x_{2n-1}\}$ or they all belong to $\{x_2, x_4, \dots, x_{2n}\}$,

3) the b_{ij}'s belong to $\{x_1, x_3, \dots, x_{2n-1}\}$ if and only if the b_{i+1j}'s belong to $\{x_2, x_4, \dots, x_{2n}\}$, and

4) $b_{ij} \neq b_{ij+1}^{\pm 1}$.

Lemma 2.3. *If $b_{ij} \in \{x_1, \dots, x_{2n-1}\}$ for $j \in \{1, \dots, m_i\}$, then $L(i_*(b_i)) > 1$. Also, the first letter of $i_*(b_i)$ is a_1 or a_1^{-1}, and the last letter of $i_*(b_i)$ is a_1 or a_1^{-1}.*

Proof. Suppose that $b_{ij} = x_{2l_j+1}$, where $l_j \in \{0, 1, \dots, n-1\}$. Then

$i_*(b_{ij}^{p_{ij}}) = (a_1a_2)^{l_j} a_1^4 (a_2a_1)^{l_j} \dots (a_1a_2)^{l_j} a_1^4 (a_2a_1)^{l_j}$, or

$i_*(b_{ij}^{p_{ij}}) = (a_2a_1)^{-l_j} a_1^{-4} (a_1a_2)^{-l_j} \dots (a_2a_1)^{-l_j} a_1^{-4} (a_1a_2)^{-l_j}$.

If $p_{ij} > 0$, and $p_{ij+1} > 0$, then

$i_*(b_{ij}b_{ij+1}) = (a_1a_2)^{l_j} a_1^4 (a_2a_1)^{l_j} (a_1a_2)^{l_{j+1}} a_1^4 (a_2a_1)^{l_{j+1}}$.

If $p_{ij} > 0$, and $p_{ij+1} < 0$, then

$i_*(b_{ij}b_{ij+1}) = (a_1a_2)^{l_j} a_1^4 (a_2a_1)^{l_j - l_{j+1}} a_1^{-4} (a_1a_2)^{-l_{j+1}}$.

Since $b_{ij} \neq b_{ij+1}$, $l_j \neq l_{j+1}$. It is easy to see that the first and last letters of $i_*(b_i)$ are a_1 or a_1^{-1}, and $L(i_*(b_i)) \geq 2$. □

Lemma 2.4. *If $b_{ij} \in \{x_2, \dots, x_{2n}\}$, then $L(i_*(b_i)) > 1$. Also, the first letter of $i_*(b_i)$ is a_2 or a_2^{-1}, and the last letter of $i_*(b_i)$ is a_2 or a_2^{-1}.*

Proof. The proof of Lemma 2.4 is similar to the proof of Lemma 2.3. □

Lemma 2.5. *S_{2n} is incompressible in H_2.*

Proof. Suppose that y is a nontrivial element of $\pi_1(S_{2n})$. Then $y = \prod_{i=1}^{m} b_i$, where b_i satisfies the above conditions. By Lemma 2.3 and Lemma 2.4, $L(i_*(y)) = \sum_{i=1}^{m} L(i_*(b_i))$. Hence $L(i_*(y)) > 1$, and S_{2n} is incompressible in H_2. □

Lemma 2.6. *S_{2n} is separating in H_2.*

Proof. By construction, $[\partial S_{2n}] = 0$ in $H_1(\partial H_2)$. Hence ∂S_{2n} is separating on ∂H_2, and S_{2n} is separating in H_2. □

Theorem 2.7. *A handlebody of genus 2 contains a separating incompressible surface S of arbitrarily high genus such that $|\partial S| = 1$.*

Proof. Let n be a positive integer, then S_{2n} is a separating incompressible surface in H_2 by the above argument. □

Remark 1. In fact a handlebody of genus 2 also contains a separating incompressible surface S of arbitrarily high genus such that $|\partial S| = 2$. For example, S_{2n-1} is a separating incompressible surface of genus $n-1$ in H_2.

Remark 2. For any positive integer n, there exist infinitely many separating incompressible surface of genus n in a handlebody of genus 2. For example, let u_1v_1 intersect D_1 in m points where $m \geq 3$. Then by the proof of Theorem 2.7, we can obtain another separating incompressible surface of genus n in H_2 by the same method as in the above construction.

Corollary 2.8. *For any integer $n > 1$, a handlebody of genus n contains a separating incompressible surface S of arbitrarily high genus.*

3. Incompressible surfaces in closed 3-manifolds of Heegaard genus 2

Let M be a compact 3-manifold with boundary. If $c_1, \dots, c_n$ are disjoint simple closed curves on ∂M, we denote by $\tau(M, \bigcup_{i=1}^{n} c_i)$ the manifold obtained by attaching 2-handles to M along disjoint regular neighborhoods of $c_1, \dots, c_n$, and $M[c_1]\dots[c_n]$ the manifold obtained by capping off possible 2-sphere components of $\partial\tau(M, \bigcup_{i=1}^{n} c_i)$. If c is a nontrivial simple closed curve on a toral component of ∂M, we denote by $M(c)$ the manifold $M[c]$. Now if $F(\neq S^2)$ is a separating incompressible closed surface in $\tau(M, \bigcup_{i=1}^{n} c_i)$, then F is also a separating incompressible surface in $M[c_1]\dots[c_n]$.

If S is a properly embedded surface in M, we denote by $\hat{S}$ the surface obtained by capping off the boundary components of S with disks in $\tau(M, \partial S)$.

Lemma 3.1. *H_2 contains no closed incompressible surface.*

Proof. Let D be a properly embedded disk in H_2 such that ∂D is nontrivial on ∂H_2. If F is a closed incompressible surface, then F may be isotoped to be disjoint from D. Hence a 3-cell contains a closed incompressible surface, a contradiction. □

Lemma 3.2. *Let M be a 3-manifold, and let J be a simple closed curve on ∂M such that $\partial M - J$ is incompressible. If M has compressible boundary, then $\tau(M, J)$ is a ∂-irreducible manifold.*

Proof. See [4, Theorem 2]. □

Let H_2 be a handlebody of genus 2, and let S be a separating incompressible surface of genus $n > 0$ in H_2 such that $|\partial S| = 1$.

Lemma 3.3. *$\tau(H_2, \partial S)$ is a ∂-irreducible 3-manifold, and $\hat{S}$ is a separating closed incompressible surface of genus n in $\tau(H_2, \partial S)$.*

Proof. Suppose that S separates H_2 into $H, H^{'}$ and ∂S separates ∂H_2 into $T, T^{'}$, such that $\partial H = S \cup T$ and $\partial H^{'} = S \cup T^{'}$.

Claim 1. $\partial H(\partial H^{'})$ is compressible in $H(H^{'})$.

Proof. If ∂H is incompressible in H, then ∂H is incompressible in H_2, contradicting Lemma 3.1. □

Claim 2. $T(T^{'})$ is incompressible in $H(H^{'})$.

Proof. If T is compressible in H, then there exists a nontrivial simple closed curve c on T such that c bounds a disk D in H. Since the genus of ∂H_2 is 2, T is a once punctured torus whose boundary is isotopic to ∂S. Hence ∂S bounds a disk in H_2, a contradiction. □

Since $\partial H(\partial H^{'})$ is compressible in $H(H^{'})$, and S and $T(T^{'})$ are incompressible in $H(H^{'})$, it follows that $\tau(H, \partial S)$ and $\tau(H^{'}, \partial S)$ are ∂-irreducible 3-manifolds by Lemma 3.2.

Since $\tau(H_2, \partial S) = \tau(H, \partial S) \cup \tau(H^{'}, \partial S)$, $\hat{S}$ is a separating incompressible closed surface in $\tau(H_2, \partial S)$. □

Since the genus of ∂H_2 is 2, $\partial\tau(H_2, \partial S)$ consists of two tori, T_1 and T_2, say.

Let M be a 3-manifold with one of component T of ∂M a torus. If r_1 and r_2 are two slopes on T, we shall denote by $\Delta(r_1, r_2)$ the minimal geometric intersection number among all the curves representing the slopes.

Lemma 3.4. *Let M be a ∂-irreducible 3-manifold with one component T of ∂M a torus, and let F be a closed incompressible surface in M which is not parallel to T. If r_1 and r_2 are two slopes on T such that F is compressible in $M(r_1)$ and $M(r_2)$, then either*

1) $\Delta(r_1, r_2) \leq 1$, *or*
2) *there exists a slope r on T such that $\Delta(r, r_1) \leq 1$ and $\Delta(r, r_2) \leq 1$.*

Proof. See [6, Theorem 1].

Corollary 3.5. *Let M be a ∂-irreducible 3-manifold with one component T of ∂M a torus. If F is a closed incompressible surface in M which is not parallel to T, then there exists a nontrivial simple closed curve c on T such that F is incompressible in $M(c)$.*

Lemma 3.6. *There exist two nonseparating simple closed curves c_1 and c_2 on ∂H_2 such that c_i is disjoint from ∂S, and $\hat{S}$ is a separating incompressible surface in $H_2[\partial S][c_1][c_2]$.*

Proof. Suppose that $\hat{S}$ separates $\tau(H_2, \partial S)$ into M_1 and M_2 such that $T_1 \subset M_1$ and $T_2 \subset M_2$. By the proof of Lemma 3.1, M_i is ∂-irreducible. Since $\hat{S}$ is not parallel to T_i in M_i, by Corollary 3.5, there exists a simple closed curve c_i $(1 \leq i \leq 2)$ on T_i such that $\hat{S}$ is incompressible in $M_i(c_i)$.

By an isotopy, we can suppose that c_i is disjoint from ∂S. Hence $\hat{S}$ is incompressible in $H_2[\partial S][c_1][c_2]$. It is easy to see that $\hat{S}$ is separating in $H_2[\partial S][c_1][c_2]$. □

Definition 3.7. Two simple closed curves α and β on ∂M are said to be coplanar if some component of $\partial M - \alpha \cup \beta$ is an annulus or a once punctured annulus.

Lemma 3.8. *Suppose that α is a nonseparating curve on ∂M. If a separating curve β on ∂M is coplanar to α, then $M[\alpha] = M[\beta][\alpha]$.*

Proof. See [5, Lemma 5.1]. □

Lemma 3.9. $H_2[\partial S][c_1][c_2] = H_2[c_1][c_2]$.

Proof. Since c_1 is coplanar to ∂S on ∂H_2, we have $H_2[\partial S][c_1][c_2] = H_2[c_1][c_2]$. □

Theorem 3.10. *For any integer $n > 0$, there exists a closed 3-manifold M of Heegaard genus 2 which contains a closed separating incompressible surface of genus n.*

Proof. Let H_2 be a handlebody of genus 2, and let S be a separating incompressible surface of genus n such that $|\partial S| = 1$. Then $H_2[c_1][c_2]$ contains a separating incompressible surface $\hat{S}$ of genus n, where c_1 and c_2 are disjoint nonseparating simple closed curves on ∂H_2 as in Lemma 3.6. Obviously the Heegaard genus of $H_2[c_1][c_2]$ is 2. □

Corollary 3.11. *Suppose that $m \geq 2$. Then for any integer $n > 0$ there exists a closed 3-manifold of Heegaard genus m which contains a closed separating incompressible surface of genus n.*

In fact there are infinitely many simple closed curves c on ∂H_2 such that $\tau(H_2, c)$ contains a closed separating incompressible surface of genus n. This is shown by the following example.

Example. Let H_2 be a handlebody of genus 2. Let S_{2m} be an incompressible surface of genus n constructed by the same method as the construction of S_{2n} (as in Section 2), such that u_1v_1 intersects D_1 in m points. Then $\hat{S}_{2m}$ is incompressible in $\tau(H_2, \partial S_{2m})$.

I am grateful to Professor R. Fintushel and the referee for their suggestions.

References

[1] C. Gordon and R. Litherland, Incompressible planar surfaces in 3-manifolds, Topology and its Appl. 18(1984), 121-144. MR **86e:**57013

[2] J. Hempel, 3-manifolds, Ann. of Math. Studies 86. Princeton Univ. Press, Princeton, New Jersey, 1976. MR **54:**3702

[3] W. Jaco, Lectures on 3-manifold Topology, Published by Amer. Math. Soc., Providence, Rhode Island, 1980. MR **81k:**57009

[4] W. Jaco, Adding a 2-handle to a 3-manifold, An application to Property R, Proc. A. M. S. 92(1984), 288-292. MR **86b:**57006

[5] M. Scharlemann and Y-Q Wu, Hyperbolic manifold and degenerating handle additions, J. Austral. Math. Soc. (series A) 55(1993), 72-89. MR **94e:**57019

[6] Y-Q Wu, Incompressibility of surfaces in syrgered 3-manifolds, Topology 31(1992), 271-280. MR **94e:**57027

Department of Mathematics, Jilin University, Changchun 130023, People's Republic of China

E-mail address: qrf@mail_jlu.edu.cn