PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 10, Pages 2875–2878
S 0002-9939(00)05366-1
Article electronically published on April 28, 2000

A REFINEMENT OF THE TORAL RANK CONJECTURE FOR 2-STEP NILPOTENT LIE ALGEBRAS

PAULO TIRAO

(Communicated by Roe Goodman)

ABSTRACT. It is known that the total (co)-homoloy of a 2-step nilpotent Lie algebra $\mathfrak{g}$ is at least $2^{|\mathfrak{z}|}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. We improve this result by showing that a better lower bound is 2^t, where $t = |\mathfrak{z}| + \left[\frac{|v|+1}{2}\right]$ and v is a complement of $\mathfrak{z}$ in $\mathfrak{g}$. Furthermore, we provide evidence that this is the best possible bound of the form 2^t.

1. Introduction

An outstanding conjecture, known as the *Toral Rank Conjecture* (TRC) claims that for any nilpotent Lie algebra $\mathfrak{g}$ (over **R** or **C**) the total (co)-homology, with trivial coefficients, satisfies the inequality

$$\left|\mathrm{H}_*(\mathfrak{g})\right| \geq 2^{|\mathfrak{z}|},$$

where $\mathfrak{z} = \text{center}(\mathfrak{g})$.

The TRC is due to S. Halperin ([5], 1987). In 1988, Deninger and Singhof [4] proved it for 2-step nilpotent Lie algebras. Besides this class, a few special cases have been added recently. For example, it was shown in [2] that the TRC holds for $\mathfrak{g}$ if its center has dimension ≤ 5 or has codimension ≤ 7.

It turns out that, in general, $2^{|\mathfrak{z}|}$ is quite a lot smaller than $|\mathrm{H}_*(\mathfrak{g})|$, especially when $|\mathfrak{z}|$ is comparatively small compared to $|\mathfrak{g}|$.

In this short note we give a new lower bound for $\left|\mathrm{H}_*(\mathfrak{g})\right|$, when $\mathfrak{g}$ is a 2-step nilpotent Lie algebra, that involves both the dimension of the center of $\mathfrak{g}$ and its codimension. Furthermore, by using existing calculations, we show that actually it is the best possible general lower bound of the form 2^t.

Precisely, we give a direct proof of the following

Theorem. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra of finite dimension over* **R** *or* **C**. *Let v be any direct complement, as vector spaces, of $\mathfrak{z} = \text{center}(\mathfrak{g})$. Then,*[1]

$$\left|\mathrm{H}_*(\mathfrak{g})\right| \geq 2^{|\mathfrak{z}| + \left[\frac{|v|+1}{2}\right]}.$$

Received by the editors August 24, 1998 and, in revised form, November 22, 1998.
2000 *Mathematics Subject Classification.* Primary 17B56, 17B30.
Key words and phrases. Homology of Lie algebras, 2-step nilpotent Lie algebras, toral rank conjecture.

[1]The right brackets denote the integral part.

2. The proof of the Theorem

We will make use of the following combinatorial result.

Lemma. *Let n be a positive integer. Then,*

$$\left|\sum_{j=0}(-1)^j\binom{n}{2j}\right| + \left|\sum_{j=0}(-1)^j\binom{n}{2j+1}\right| = 2^{\left[\frac{n+1}{2}\right]}.$$

Proof. Let $P=\sum_{j=0}(-1)^j\binom{n}{2j}$ and $Q=\sum_{j=0}(-1)^j\binom{n}{2j+1}$. Since $(1+i)^n=P+iQ$ and $(1-i)^n=P-iQ$, then $P=\frac{(1+i)^n+(1-i)^n}{2}$ and $Q=i^{-1}\frac{(1+i)^n-(1-i)^n}{2}$. Therefore,

$$|P| = \begin{cases} 2^{n/2}, & \text{if } n\equiv 0 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 1 \mod 4;\\ 0, & \text{if } n\equiv 2 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 3 \mod 4;\end{cases} \quad\text{and}\quad |Q| = \begin{cases} 0, & \text{if } n\equiv 0 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 1 \mod 4;\\ 2^{n/2}, & \text{if } n\equiv 2 \mod 4;\\ 2^{(n-1)/2}, & \text{if } n\equiv 3 \mod 4;\end{cases}$$

from which the lemma follows. □

Proof of the Theorem. The homology of $\mathfrak{g}$ is the homology of the Koszul complex $\left(\bigwedge\mathfrak{g},\partial\right)$.[2]

Put $\mathfrak{g}=v\oplus\mathfrak{z}$. Hence, we can write $\bigwedge\mathfrak{g}=\bigwedge v\otimes\bigwedge\mathfrak{z}$. It is straightforward to see that

$$\partial : \bigwedge^{p} v\otimes\bigwedge^{q}\mathfrak{z} \longrightarrow \bigwedge^{p-2} v\otimes\bigwedge^{q+1}\mathfrak{z}.$$

Therefore, it follows that the complex $\left(\bigwedge\mathfrak{g},\partial\right)$ is the direct sum of an *even* and an *odd* subcomplex, precisely $\left(\bigwedge^{2p} v\otimes\bigwedge\mathfrak{z},\partial\right)$ and $\left(\bigwedge^{2p+1} v\otimes\bigwedge\mathfrak{z},\partial\right)$. Accordingly, the homology of the Koszul complex is the sum of the homologies of each of the even and odd subcomplexes.

It is well known that if $\left(\mathcal{C}=(C_i),\partial\right)$ is a finite complex of finite dimensional vector spaces, then

$$\left|\mathrm{H}_*(\mathcal{C})\right| \geq \left|\sum_i(-1)^i\dim\left(C_i\right)\right|.$$

By applying this to each of the even and odd subcomplexes we get that

$$\begin{aligned}\left|\mathrm{H}_*(\mathfrak{g})\right| &\geq \left|\sum_{j=0}(-1)^j\dim\left(\bigwedge^{2j} v\otimes\bigwedge\mathfrak{z}\right)\right| + \left|\sum_{j=0}(-1)^j\dim\left(\bigwedge^{2j+1} v\otimes\bigwedge\mathfrak{z}\right)\right|\\ &= 2^{|\mathfrak{z}|}\left|\sum_{j=0}(-1)^j\binom{|v|}{2j}\right| + 2^{|\mathfrak{z}|}\left|\sum_{j=0}(-1)^j\binom{|v|}{2j+1}\right|.\end{aligned}$$

Now the Theorem follows from the Lemma. □

Remark. The bound obtained in the Theorem is the best possible, of the form 2^t (t an integer), that can be given in general. In fact, among the few available computations there are examples that support this claim.

[2] $\partial(x_1\wedge\ldots\wedge x_p)=\sum_{i<j}(-1)^{i+j+1}[x_i,x_j]\wedge x_1\wedge\ldots\wedge\widehat{x_i}\wedge\ldots\wedge\widehat{x_j}\wedge\ldots\wedge x_p$.

Example 1. Let $\mathfrak{g}_2$ denote the Lie algebra with basis $\{x_1, x_2, y_1, y_2, z\}$ and non-zero brackets $[z, x_i] = y_i$ for each $1 \leq i \leq 2$. It was shown in [1] that

$$b_0 = 1, \quad b_1 = 3, \quad b_2 = 6, \quad b_3 = 6, \quad b_4 = 3, \quad b_5 = 1,$$

where $b_i = |\mathrm{H}^i(\mathfrak{g}_2)|$. Therefore, the total cohomology of $\mathfrak{g}_2$ is equal to 20 and our bound is $2^{2+2} = 16$.

Example 2. Let $\mathfrak{g}$ denote the Lie algebra with basis $\{a, b, c, d, e, f, g\}$ and non-zero brackets $[a, b] = e$, $[b, d] = g$, $[c, d] = e$ and $[a, c] = f$. This Lie algebra appears as $3, 7_D$ in Seeley's classification [6]. In [3] one can explicitly find that

$$b_0 = 1, \quad b_1 = 4, \quad b_2 = 11, \quad b_3 = 14, \quad b_4 = 14, \quad b_5 = 11, \quad b_6 = 4, \quad b_7 = 1,$$

where $b_i = |\mathrm{H}^i(\mathfrak{g})|$. Hence $|\mathrm{H}^*(\mathfrak{g})| = 60$, while our bound is $2^{3+2} = 32$.

Example 3. We consider here a family of examples. For each $r \geq 2$ let E be an r-dimensional vector space; then $\mathfrak{g}_r = E \oplus \bigwedge^2 E$ is the rank r 2-step free nilpotent Lie algebra. For $e, f \in E$, $[e, f] = e \wedge f$, these being all the non-zero brackets.

Their homology has been computed by Sigg [7]. Using his result one can compute, for each r, the total homology for the rank r algebra. With a simple computer program written in Maple V we have done it for small r's. On the other hand, it is clear that our bound is $2^{\binom{r}{2} + \left[\frac{r+1}{2}\right]}$. In the following table we show the results for $2 \leq r \leq 13$. $\left(t = \binom{r}{2} + \left[\frac{r+1}{2}\right]\right)$

r	$\lvert\mathfrak{g}_r\rvert$	Total homology	t	2^t
2	3	6	2	4
3	6	36	5	32
4	10	420	8	256
5	15	9800	13	8192
6	21	452760	18	262144
7	28	4.1835024×10^{7}	25	3.3554432×10^{7}
8	36	7.691667984×10^{9}	32	4.294967296×10^{9}
9	45	$2.828336198688\times 10^{12}$	41	$2.199023255552\times 10^{12}$
10	55	$2.073619375892064\times 10^{15}$	50	$1.125899906842624\times 10^{15}$
11	66	$3.040584296923128384\times 10^{18}$	61	$2.305843009213693952\times 10^{18}$
12	78	$8.898500292240756664896\times 10^{21}$	72	$4.722366482869645213696\times 10^{21}$
13	91	$5.208427046810518523791884 8\times 10^{25}$	85	$3.8685626227668133590597632\times 10^{25}$

Acknowledgments

We thank the referee for his illuminating comments.

References

[1] Armstrong G., Cairns G. and Jessup B., *Explicit Betti numbers for a family of nilpotent Lie algebras*, Proc. Amer. Math. Soc. **125** (1997), 381–385. MR **97d:**17013

[2] Cairns G. and Jessup B., *New bounds on the Betti numbers of nilpotent Lie algebras*, Comm. in Algebra **25** (1997), 415–430. MR **98a:**17032

[3] Cairns G., Jessup B., and Pitkethly J., *On the Betti numbers of nilpotent Lie algebras of small dimension*, Prog. in Math.: Integrable systems and foliations (Albert C. et al., eds.), vol. 145, Birkhauser, 1997, pp. 19–31. MR **98c:**17018

[4] Deninger Ch. and Singhof W., *On the cohomology of nilpotent Lie algebras*, Bull. Soc. Math. France **116** (1988), 3–14. MR **90c:**17023

[5] Halperin S., *Le complexe de Koszul en algèbre et topologie*, Ann. Inst. Fourier **37** (1987), 77–97. MR **89d:**55040

[6] Seeley C., *7-dimensional nilpotent Lie algebras*, Trans. Amer. Math. Soc. **335** (1993), 479–496. MR **93d:**17015

[7] Sigg S., *Laplacian and homology of free 2-step nilpotent Lie algebras*, J. Algebra **185** (1996), 144–161. MR **97i:**17014

International Centre for Theoretical Physics (ICTP), Trieste, Italy

Facultad de Matemática, Astronomía y Física, Córdoba, Argentina

Current address: Heinrich-Heine-Universität, Mathematisches Institut, 40225 Düsseldorf, Germany

E-mail address: `ptirao@bart.cs.uni-duesseldorf.de`

E-mail address: `Paulo.Tirao@FamaF.uncor.edu.ar`