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IDEALS WITHOUT CCC AND WITHOUT PROPERTY (M)

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ABSTRACT. We prove a strong version of a theorem of Balcerzak-Roslanowski-Shelah by showing, in ZFC, that there exists a simply definable Borel σ-ideal for which both the ccc and property (M) fail. The proof involves Polish group actions.

Definition. A *Borel ideal* is an ideal I on a Polish space X with the following property: for all $A \in I$, there exists a Borel subset B of X such that $A \subset B$ and $B \in I$. We consider three types of properties for a Borel ideal I on a Polish space X.

(1) *I has a Π^1_n definition* if the following set is Π^1_n:

$$\mathcal{C}_I = \{c \in 2^\omega : c \text{ is a Borel code for a subset } B_c \text{ of } X \text{ and } B_c \in I\}.$$

(2) For κ a cardinal, *I satisfies the κ-cc* if there does not exist a family of κ I-almost disjoint Borel sets that are not in I. The ω_1-cc is usually called the *ccc*.

(3) I satisfies *property* (M) if there exists a Borel-measurable function $f : X \to 2^\omega$ with $f^{-1}(y) \notin I$ for all $y \in 2^\omega$.

Property (M) was introduced in Balcerzak [1]. Obviously, an ideal satisfying property (M) violates the ccc. Both the above paper and the later paper of Balcerzak-Roslanowski-Shelah [2] are concerned with circumstances in which it is possible for both the ccc and property (M) to fail. We refer the reader to these two references for information on these properties—and several other properties—of ideals. The latter paper contains the following result.

Theorem 1 (Balcerzak-Roslanowski-Shelah [2, 5.6]). *Assume that either CH fails or every $\mathbf{\Delta}^1_2$ set of reals has the Baire property. Then there exists a Borel σ-ideal I^*, containing all singletons, such that:*

(a) *I^* has a Π^1_2 definition;*
(b) *the ccc fails for I^*;*
(c) *property (M) fails for I^*;*
(d) *I^* satisfies the ω_2-cc.*

Andrzej Roslanowski presented this result in a seminar talk at Ohio State University in 1993, and asked whether or not it is provable in ZFC. I thank him for bringing this question to my attention, and I thank Ohio State for their support.

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The purpose of this paper is to prove Theorem 2, below, which improves Theorem 1 in two ways. First, we provide a positive answer to Roslanowski's question by proving the result in ZFC. Second, we strengthen part (a) of the theorem from Π^1_2 to Π^1_1.

Theorem 2. *There exists a Borel σ-ideal I, containing all singletons, such that:*
(a) *I has a Π^1_1 definition;*
(b) *the ccc fails for I;*
(c) *property (M) fails for I;*
(d) *I satisfies the ω_2-cc.*

Proof. Let X be the Polish space $2^{(\omega\times\omega)}$. Let L be the language with one binary relation symbol. We view X as the space of codes for L-structures with universe ω, by identifying a relation with its characteristic function. That is, $x \in X$ encodes the L-structure $\mathfrak{A}_x = \langle \omega, \leq_x \rangle$, where $m \leq_x n$ iff $x(m,n) = 1$. Let S_∞ denote the group of permutations of ω, topologized as a subspace of ω^ω. As this subspace is G_δ, S_∞ is a Polish topological group (see Kechris [6, 3.C and 9.B]). The logic action $a_L : S_\infty \times X \to X$ is defined by

$$a_L(g,x) = y \text{ iff } (\forall m,\ n \in \omega)[y(m,n) = 1 \Longleftrightarrow x(g^{-1}(m), g^{-1}(n)) = 1].$$

That is, $a_L(g,x) = y$ iff the permutation $g : \omega \to \omega$ is an isomorphism of $\mathfrak{A}_x$ onto $\mathfrak{A}_y$. This is a continuous action. The orbit equivalence relation is isomorphism (of L-structures encoded by elements of X). For information on logic actions, see Becker-Kechris [4, 2.5].

Let $\mathcal{O}$ be any a_L-orbit. A subset N of X is called $\mathcal{O}$-*negligible* iff for some (equivalently, for any) $x \in \mathcal{O}$, the following subset of S_∞ is meager: $\{g : a_L(g,x) \in N\}$. Let $J_\mathcal{O}$ be the set

$$\{N \subset X : \text{There exists a Borel set } B \subset X \text{ such that } N \subset B \text{ and } B \text{ is } \mathcal{O}\text{-negligible}\}.$$

Clearly $J_\mathcal{O}$ is a Borel σ-ideal on X. Since the meager ideal on S_∞ satisfies the ccc, so does $J_\mathcal{O}$. And if $\mathcal{O}$ is an uncountable orbit, then $J_\mathcal{O}$ contains all singletons.

Let η denote the order-type of the rational numbers. We define an a_L-invariant subset S of X as follows:

$$S = \{x : \text{There exists a countable admissible ordinal } \alpha \text{ such that } \mathfrak{A}_x \text{ is a linear ordering of order-type } \alpha(1+\eta)\}.$$

Let I be the set

$$\{N \subset X : \text{There exists a Borel set } B \subset X \text{ such that } N \subset B \text{ and such that for every orbit } \mathcal{O} \subset S,\ B \text{ is } \mathcal{O}\text{-negligible}\}.$$

Note that every orbit in S is uncountable. It therefore follows easily from the properties of the $J_\mathcal{O}$'s that I is a Borel σ-ideal on X containing all singletons. It remains to be shown that I satisfies conditions (a)–(d) of this theorem.

(a) By definition of I and $\mathcal{C}_I$, for all $c \in 2^\omega$, $c \in \mathcal{C}_I$ iff:

$$[c \text{ is a Borel code for a subset } B_c \text{ of } X] \ \&\ (\text{for all } x \in X)\ [x \notin S \text{ or (for a comeager set of } g \in S_\infty)(a_L(g,x) \notin B_c)].$$

Friedman [5] proved that S is a Σ^1_1 set. Hence routine quantifier-counting, applied to the above formula, demonstrates that $\mathcal{C}_I$ is a Π^1_1 set. (See Kechris [6, 29.22 and 35.B] or Moschovakis [8, 4F.19 and 7B.1].)

(b) Every a_L-orbit is a Borel set in X. This fact is a special case of a theorem of Miller (see Kechris [6, 15.14]); it also follows from the theorem of Scott (see Keisler [7, Chapter 2]) that countable L-structures are characterized up to isomorphism by an $L_{\omega_1\omega}$ sentence. Clearly S contains uncountably many orbits. So the family of all orbits in S witnesses that the ccc fails.

(d) Suppose $\{B_\zeta : \zeta < \omega_2\}$ is a family of I-almost disjoint Borel subsets of X, and for all ζ, $B_\zeta \notin I$. Then for each ζ, there is an orbit $\mathcal{O}_\zeta \subset S$ such that B_ζ is not $\mathcal{O}_\zeta$-negligible. As S contains only ω_1 orbits, there are a fixed orbit $\mathcal{O} \subset S$ and a set $W \subset \omega_2$ of cardinality ω_2 such that for all $\zeta \in W$, B_ζ is not $\mathcal{O}$-negligible. This contradicts the fact that $J_\mathcal{O}$ satisfies the ccc.

(c) Suppose that I does satisfy property (M). Let $f : X \to 2^\omega$ be a Borel function such that for all $y \in 2^\omega$, $f^{-1}(y) \notin I$. For $x \in X$, let $\mathcal{O}(x)$ denote the orbit of x. Define $R \subset 2^\omega \times X$ as follows:

$$R = \{(y,x) : x \in S \ \&\ f^{-1}(y) \text{ is not } \mathcal{O}(x)\text{-negligible}\}.$$

By definition of f and I, every point y in 2^ω is in the domain of R. By definition of negligible, for all $(y,x) \in 2^\omega \times X$, $(y,x) \in R$ iff:

$$x \in S \ \&\ (\text{for a nonmeager set of } g \in S_\infty)(a_L(g,x) \in f^{-1}(y)).$$

The above formula shows that R is $\boldsymbol{\Sigma}^1_1$ (cf. proof of part (a)). The Jankov-von Neumann Theorem (see Kechris [6, 29.9] or Moschovakis [8, 4E.9]) states that $\boldsymbol{\Sigma}^1_1$ relations admit measurable selections. Hence there is a Lebesgue measurable function $F : 2^\omega \to X$ such that for all $y \in 2^\omega$, $F(y) \in S$ and $f^{-1}(y)$ is not $\mathcal{O}(F(y))$-negligible. Since the ideals $J_\mathcal{O}$ satisfy the ccc, for any orbit $\mathcal{O}$, $F^{-1}[\mathcal{O}]$ is countable. Therefore, to complete the proof, it will suffice to show that there exists an uncountable set $Q \subset 2^\omega$ such that $F[Q]$ intersects only countably many orbits of S.

Since F is Lebesgue measurable, there exists a subset of 2^ω of positive measure on which F is continuous; hence there is a nonempty perfect set $P \subset 2^\omega$ such that $F \restriction P$ is continuous. Let $r \in 2^\omega$ be such that P is a recursive-in-r presented Polish space (i.e., P is the set of branches of a recursive-in-r pruned tree) and $F \restriction P$ is a recursive-in-r function from P into X. For points z in either of the spaces 2^ω or X $(= 2^{(\omega\times\omega)})$, let ω_1^z denote, as usual, the least ordinal not recursive-in-z. Let

$$H = \{\mathcal{O} : \mathcal{O} \text{ is an } a_L\text{-orbit and there exists an } x \in \mathcal{O} \text{ such that } \omega_1^x \le \omega_1^r\}.$$

Let $Q = \{y \in P : \omega_1^{\langle r,y\rangle} = \omega_1^r\}$. Then, as proved in Moschovakis [8, 4F.21], Q is uncountable. For any $y \in Q$, $\omega_1^{F(y)} \le \omega_1^{\langle r,y\rangle} = \omega_1^r$; that is, $F[Q] \subset \cup H$. But H contains only countably many orbits of S. This is so because, by definition of S, for all but countably many orbits $\mathcal{O} \subset S$, every point x in $\mathcal{O}$ encodes a linear ordering $\mathfrak{A}_x$ of ω with the property that for some $n \in \omega$, the initial segment of $\mathfrak{A}_x$ below n has order-type ω_1^r. □

Remarks. We conclude this paper with four remarks regarding variations on the proof of Theorem 2.

(1) The argument involving Lebesgue measure can be replaced by an analogous argument involving Baire category.

(2) Let $S^* = \{x \in X : \mathfrak{A}_x \text{ is a wellordering}\}$. Suppose that we define an ideal I^* from S^* in the same way that I was defined from S in the proof of Theorem 2. In this case, the proof of Theorem 2 would give Theorem 1. The formula in

the proof of (a) would be Π^1_2 rather than Π^1_1. Similarly, the set R, defined in the proof of (c), would be $\mathbf{\Pi}^1_1$ rather than $\mathbf{\Sigma}^1_1$. It is not provable in ZFC that a $\mathbf{\Pi}^1_1$ relation can be uniformized by a function which is measurable or which has the Baire property. But (in ZFC) a $\mathbf{\Pi}^1_1$ relation does have a $\mathbf{\Delta}^1_2$ uniformization; so if $\mathbf{\Delta}^1_2$ sets are measurable, the proof goes through; and if $\mathbf{\Delta}^1_2$ sets have the Baire property, then, by Remark (1), the proof also goes through, giving us Theorem 1. This is not the original Balcerzak-Roslanowski-Shelah [2] proof of Theorem 1, but the two proofs have a great deal in common.

(3) Let G' be a Polish group, let X' be a Polish space, let $a' : G' \times X' \to X'$ be a continuous action and let $S' \subset X'$ be an a'-invariant $\mathbf{\Sigma}^1_1$ set containing no countable orbits. Suppose that S' contains uncountably many a'-orbits but S' does not have a perfect subset consisting of a'-inequivalent points. (Assuming $\neg$ CH, this is equivalent to saying that S' has exactly ω_1 orbits.) We can then define an ideal I' on X' from G', a' and S' in the same way that the ideal I on X was defined from S_∞, a_L and S in the proof of Theorem 2. This ideal I' would also satisfy Theorem 2. The second last sentence in the proof of Theorem 2 is still true for S', although harder to prove than in the special case where $S' = S$. A proof can be found in Becker [3, §3]. Everything else in the proof of Theorem 2 works for any such I'.

(4) Let G', X' and a' be as in Remark (3), and let $S'' \subset X'$ be an arbitrary a'-invariant set. Let I'' be the ideal on X' defined from G', a' and S'' in the above manner. Suppose that S'' *does* have a perfect subset consisting of a'-inequivalent points. Then it can be shown that I'' satisfies property (M).

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