

AUSLANDER-REITEN SEQUENCES UNDER BASE FIELD EXTENSION

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Dedicated to Professor Helmut Lenzing on the occasion of his sixtieth birthday

ABSTRACT. We investigate the behaviour of Auslander-Reiten sequences of modules over a finite dimensional algebra over a field k under base field extension. It is proved that an Auslander-Reiten sequence splits into a direct sum of Auslander-Reiten sequences provided the extension is separable in the sense of MacLane.

1. INTRODUCTION

Throughout this paper R denotes a finite dimensional algebra over a field k and $R^{(K)} = R \otimes_k K$, where $k \subseteq K$ is a field extension. We investigate the behaviour of the functor

$$(1.1) \quad (-) \otimes_k K : \text{mod}(R) \longrightarrow \text{mod}(R^{(K)})$$

on almost split sequences and irreducible maps in the category $\text{mod}(R)$ of right finitely generated R -modules. The main result is Theorem 3.8 asserting that after applying the functor (1.1) to a minimal almost split map in the category $\text{mod}(R)$ we obtain a direct sum of minimal almost split maps in the category $\text{mod}(R^{(K)})$ if the extension $k \subseteq K$ is separable in the sense of MacLane [4], [3]. Using this result we obtain an alternative proof of the result of Jensen and Lenzing [4], that the algebra $R^{(K)}$ is of finite representation type provided R is also and the extension $k \subseteq K$ is MacLane separable.

In Section 4 we consider the following problem: determine components \mathcal{D} of the Auslander-Reiten quiver of $\text{mod}(R^{(K)})$ such that if \mathcal{D} contains a module defined over k , then all modules in \mathcal{D} are also defined over k . We prove in Proposition 4.12 that stable and semi-stable infinite components containing a cycle of irreducible maps have the above property. We also make some observations of this kind in the case when the algebra R is hereditary.

Throughout this paper given any Artin algebra A (see [1]) we denote by $\text{rad } A$ the Jacobson radical of A and by A^{op} the algebra opposite to A . All modules considered are right finitely generated. By $\text{mod}(A)$ we denote the category of right finitely generated A -modules. Given an A -module X we denote by $\text{rad } X$ and $\text{soc } X$ the radical and the socle of X respectively. By Γ_A we denote the Auslander-Reiten

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quiver of A , $\tau_A = DTr$, $\tau_A^- = TrD$ are the Auslander-Reiten translates in $\text{mod}(A)$. By a component of Γ_A we mean a connected component of Γ_A . We often identify vertices of Γ_A with A -modules. An arrow $X \rightarrow Y$ in Γ_A has valuation (a, b) if a is the multiplicity of X as a direct summand in the domain E of the minimal right almost split morphism $E \rightarrow Y$ and b is the multiplicity of Y as a direct summand in the codomain E' of the minimal left almost split morphism $X \rightarrow E'$. For further notation related with Auslander-Reiten quiver and almost split sequences we refer to [1].

Following [6] we denote by ${}_s\Gamma_A$ (resp. ${}_l\Gamma_A, {}_r\Gamma_A$) the stable (resp. left stable, right stable) part of Γ_A obtained by removing τ_A -orbits of projective and injective modules (resp. projective modules, injective modules). A connected component of ${}_s\Gamma_A$ (resp. ${}_l\Gamma_A, {}_r\Gamma_A$) is called a stable (resp. left stable, right stable) component of Γ_A . By a semi-stable component we mean a left stable or right stable component.

If \mathcal{C} is a connected subquiver of Γ_A and X and Y are vertices of \mathcal{C} , then the **distance** from X to Y in \mathcal{C} is by the definition the minimal length $\text{dist}_{\mathcal{C}}(X, Y)$ of a (non-oriented) path in \mathcal{C} connecting X with Y . The distance is zero if and only if $X \cong Y$.

Let $k \subseteq K$ be an arbitrary field extension. Given an embedding $u : X \hookrightarrow Y$ of k -vector spaces we shall identify $X \otimes_k K$ with the image of $u \otimes K$. For simplicity we shall write $X^{(K)}$ instead of $X \otimes_k K$. If X has an additional structure of an algebra or a module, then $X^{(K)}$ has the structure induced by the structure of X .

2. PRELIMINARY FACTS

In this section we collect some observations concerning the behaviour of the category $\text{mod}(R)$ under the action of the functor (1.1). Note that this functor is exact. We shall say that an $R^{(K)}$ -module X is **defined over k** provided X is isomorphic to $Y^{(K)}$ for an R -module Y .

Lemma 2.1. *If P is a projective R -module, then $P^{(K)}$ is a projective $R^{(K)}$ -module. Each indecomposable projective $R^{(K)}$ -module is a direct summand of $P^{(K)}$ for some projective R -module P .*

The proof is obvious. □

Lemma 2.2. (a) *For any R -modules X, Y and $i \geq 0$ the canonical homomorphism*

$$\text{Ext}_R^i(X, Y) \otimes_k K \longrightarrow \text{Ext}_{R^{(K)}}^i(X^{(K)}, Y^{(K)})$$

is an isomorphism. The isomorphisms are natural with respect to homomorphisms $X' \rightarrow X, Y \rightarrow Y'$.

(b) *For any right R -module X and left R -module Y and $i \geq 0$ the canonical homomorphism*

$$\text{Tor}_R^i(X, Y) \otimes_k K \longrightarrow \text{Tor}_{R^{(K)}}^i(X^{(K)}, Y^{(K)})$$

is an isomorphism. The isomorphisms are natural with respect to homomorphisms $X \rightarrow X', Y \rightarrow Y'$.

Proof. (a) The statement follows easily for $X = R$ and then for X projective. If X is arbitrary, consider the projective resolution

$$\mathbf{P}_* : \dots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \longrightarrow 0$$

of X . By applying to \mathbf{P}_* the functor $(-)\otimes_k K$ we obtain by Lemma 2.1 a projective resolution of $X^{(K)}$. The complexes $\text{Hom}_R(\mathbf{P}_*, Y)^{(K)}$ and $\text{Hom}_{R^{(K)}}(\mathbf{P}_*^{(K)}, Y^{(K)})$ are isomorphic. The assertion (a) follows since the functor $(-)\otimes_k K : \text{mod}(k) \rightarrow \text{mod}(K)$ is exact.

The proof of (b) is analogous. □

Lemma 2.3. $(R^{op})^{(K)} \cong (R^{(K)})^{op}$ and for any R -module X there exists a natural isomorphism $\text{Hom}_k(X, k) \otimes_k K \cong \text{Hom}_K(X^{(K)}, K)$ of $(R^{(K)})^{op}$ -modules.

The proof is routine. □

The following corollary is a direct consequence of Lemmata 2.3 and 2.1.

Corollary 2.4. *If E is an injective R -module, then $E^{(K)}$ is an injective $R^{(K)}$ -module. Each indecomposable injective $R^{(K)}$ -module is a direct summand of $E^{(K)}$ for some injective R -module E .* □

Lemma 2.5. *Let X, Y be R -modules.*

(a) *If X is indecomposable, then X is a direct summand of Y if and only if $X^{(K)}$ and $Y^{(K)}$ have a common nonzero direct summand.*

(b) *$X \cong Y$ if and only if $X^{(K)} \cong Y^{(K)}$.*

(c) *If X is indecomposable and $X^{(K)} \cong X_1 \oplus X_2$ with X_1, X_2 nonzero, then X_1 and X_2 are not defined over k .*

Proof. (a) The “only” part is obvious. For the proof of the converse implication we assume that $X^{(K)}$ and $Y^{(K)}$ have a common nonzero direct summand. It follows that there exist $R^{(K)}$ -homomorphisms $f : X^{(K)} \rightarrow Y^{(K)}$ and $g : Y^{(K)} \rightarrow X^{(K)}$ such that gf is a nonzero idempotent in $\text{End}_{R^{(K)}}(X^{(K)})$. It follows from Lemma 2.2 that there exist $f_i \in \text{Hom}_R(X, Y)$, $g_j \in \text{Hom}_R(Y, X)$, $\lambda_i, \mu_j \in K$, $i = 1, \dots, r$, $j = 1, \dots, s$, such that $f = \sum_{i=1}^r f_i \otimes \lambda_i$ and $g = \sum_{j=1}^s g_j \otimes \mu_j$. We treat the canonical isomorphisms from Lemma 2.2 as identities. Then $gf = \sum_{i=1}^r \sum_{j=1}^s g_j f_i \otimes \lambda_i \mu_j$. Since this is a nonzero idempotent and the ring $\text{End}_R(X)$ is local, it follows that for some i, j the homomorphism $g_j f_i$ is invertible in $\text{End}_R(X)$. Thus X is a direct summand of Y .

Statement (b) follows directly from (a).

(c) If X_1 is defined over k , say $X_1 \cong Y^{(K)}$ for an R -module Y , then by (a) X is a direct summand of Y . Comparing the lengths of X and Y we obtain $X \cong Y$ and $X_2 = 0$, a contradiction. □

3. MACLANE SEPARABLE EXTENSIONS AND ALMOST SPLIT SEQUENCES

A crucial role in our considerations will be played by field extensions separable in the sense of MacLane; see Definition 3.1 below. The significance of this class of extensions for the representation theory of algebras was observed in [4].

Definition 3.1 ([3, Chapter IV, Definition 3]). A field extension $k \subseteq K$ is **MacLane separable** if $\text{char } k = 0$ or $\text{char } k = p > 0$ and K is linearly disjoint from $k^{p^{-1}}$ over k , that is, the natural homomorphism $K \otimes_k k^{p^{-1}} \rightarrow K k^{p^{-1}}$ is bijective.

Here $k^{p^{-1}}$ is the subfield of the algebraic closure of k consisting of elements x such that $x^p \in k$. We consider K and $k^{p^{-1}}$ as subfields of the algebraic closure \overline{K} of K . □

Lemma 3.2. *If $k \subseteq K$ is a MacLane separable field extension and A is a semisimple finite dimensional k -algebra, then $A^{(K)}$ is a semisimple K -algebra.*

Proof. The assertion follows from a result of Jensen and Lenzing [4] that the global dimension of a k -algebra is preserved under a MacLane separable extension of the base field k . However for convenience of the reader we present an alternative proof of semisimplicity of $A^{(K)}$ which does not use model theoretical arguments.

It is enough to prove the assertion in the case of A being a division algebra. If A is a field, then by [3, Chapter IV, Theorem 23] we know that $A^{(K)}$ has no nonzero nilpotent elements. Since $A^{(K)}$ has finite dimension over K , it follows that $A^{(K)}$ is semisimple.

If A is not a field, then by [2, Theorem 4.2.1] there exists a maximal subfield $k \subseteq L \subseteq A$ such that $L \otimes_{Z(A)} A \cong M_n(L)$, the full $n \times n$ matrix algebra with coefficients in L , where $n = [A : Z(A)]$ and $Z(A)$ is the center of A . Hence

$$L \otimes_{Z(A)} A \otimes_k K \cong M_n(L \otimes_k K).$$

Since $L \otimes_k K$ is semisimple, it follows that the algebra $M_n(L \otimes_k K)$ has no nonzero two-sided nilpotent ideals so $A \otimes_k K$ has no nonzero two-sided nilpotent ideals. Thus $A \otimes_k K$ is a semisimple K -algebra. □

An immediate consequence of the above lemma is the following fact.

Lemma 3.3. (a) $(\text{rad } R)^{(K)} \subseteq \text{rad } R^{(K)}$,
 (b) $(\text{rad } R)^{(K)} = \text{rad } R^{(K)}$ if $k \subseteq K$ is MacLane separable extension.

Proof. Statement (a) follows immediately since $\text{rad } R$ and consequently $(\text{rad } R)^{(K)}$ are nilpotent ideals in R and $R^{(K)}$ respectively. To prove (b) it is enough to observe that it follows from Lemma 3.2 that the algebra $R^{(K)}/(\text{rad } R)^{(K)}$ is semisimple if $k \subseteq K$ is a MacLane separable extension. □

Remark 3.4. The assumption that $k \subseteq K$ is MacLane separable is essential. Consider the following example from [3]. Let k be a field of characteristic 2 containing an element t being not a square of an element of k . Let $K = k(\sqrt{t}) = k[S]/(S^2 - t)$. Then $K \otimes_k K$ is commutative but contains a nonzero nilpotent element $1 \otimes s - s \otimes 1$, where s is the residue class of the variable S . Hence it is not semisimple. □

Corollary 3.5. (a) *If X is an R -module, then $(\text{rad } X)^{(K)} \subseteq \text{rad } X^{(K)}$ and if the extension $k \subseteq K$ is MacLane separable, then $(\text{rad } X)^{(K)} = \text{rad } X^{(K)}$.*

(b) *If X is an R -module, then $\text{soc } X^{(K)} \subseteq (\text{soc } X)^{(K)}$ and if the extension $k \subseteq K$ is MacLane separable, then $\text{soc } X^{(K)} = (\text{soc } X)^{(K)}$.*

(c) *If $P \rightarrow X$ is a minimal projective cover of X in $\text{mod}(R)$, then the induced homomorphism $P^{(K)} \rightarrow X^{(K)}$ is a minimal projective cover of $X^{(K)}$ in $\text{mod}(R^{(K)})$.*

(d) *If $X \rightarrow E$ is a minimal injective envelope of X in $\text{mod}(R)$, then the induced homomorphism $X^{(K)} \rightarrow E^{(K)}$ is a minimal injective envelope of $X^{(K)}$ in $\text{mod}(R^{(K)})$.*

Proof. The proof of (a) follows from the well-known fact that for a finitely generated module X over a finite dimensional algebra R we have $\text{rad } X = X \text{rad } R$. Statement (b) follows dually whereas (c) and (d) are direct consequences of (a), (b), Lemma 2.1 and Corollary 2.4. □

It follows immediately from the above corollary that if

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

is a minimal projective presentation of X in $\text{mod}(R)$, then the induced sequence

$$P_1^{(K)} \longrightarrow P_0^{(K)} \longrightarrow X^{(K)} \longrightarrow 0$$

is a minimal projective presentation of $X^{(K)}$ in $\text{mod}(R^{(K)})$. This observation together with Lemmata 2.1, 2.2 and 2.3 yields the following corollary.

Corollary 3.6. *For any X in $\text{mod}(R)$ there exist isomorphisms $(\tau_R X)^{(K)} \cong \tau_{R^{(K)}} X^{(K)}$ and $(\tau_R^- X)^{(K)} \cong \tau_{R^{(K)}}^- X^{(K)}$. \square*

Now we turn to the main question of this section. We shall deal with almost split sequences in $\text{mod}(R)$ and apply the functor (1.1) to them. Let us precede the main theorem by the following lemma.

Lemma 3.7. *Let A and B be rings with 1 and let S be an A - B -bimodule which is simple as a left A module and as a right B -module. Given a natural number $n \in \mathbb{N}$ let A_n and B_n be the full $n \times n$ matrix rings with coefficients in A and B respectively. Consider $S_n = S \otimes_B B_n \cong A_n \otimes_A S$ as an A_n - B_n -bimodule with the obvious structure. Then for any element $s = [s_{ij}] \in S_n$ there exist invertible elements $a \in A_n$ and $b \in B_n$ such that asb is a diagonal element of S_n , that is, $asb = [s'_{ij}]$, where $s'_{ij} = 0$ for $i \neq j$.*

Proof. The proof follows from the observation that we can diagonalize the element $[s_{ij}]$ by elementary row and column transformations. \square

Theorem 3.8. *Let $X \rightarrow Y$ be a minimal left (resp. right) almost split homomorphism in $\text{mod}(R)$ and the extension $k \subseteq K$ be MacLane separable. Then the induced homomorphism $X^{(K)} \rightarrow Y^{(K)}$ is isomorphic to a direct sum of minimal left (resp. right) almost split homomorphisms in $\text{mod}(R^{(K)})$. In particular, if*

$$0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0$$

is an Auslander-Reiten sequence in $\text{mod}(R)$ and $Y^{(K)} \cong \bigoplus_{i=1}^r Z_i$ with indecomposable modules Z_i , then the induced exact sequence

$$0 \longrightarrow X^{(K)} \longrightarrow E^{(K)} \longrightarrow Y^{(K)} \longrightarrow 0$$

is isomorphic to the direct sum of Auslander-Reiten sequences

$$0 \longrightarrow \tau_{R^{(K)}} Z_i \longrightarrow F_i \longrightarrow Z_i \longrightarrow 0$$

for $i = 1, \dots, r$.

Proof. Assume first that Y is projective and not simple. Then the embedding $\text{rad } Y \rightarrow Y$ is the minimal right almost split morphism. Thus the assertion follows by Corollary 3.5 (a). In case of a minimal left almost split morphism starting from an injective non-simple module X we proceed dually.

Now consider the case of Y non-projective and let

$$e : 0 \longrightarrow \tau_R Y \longrightarrow E \longrightarrow Y \longrightarrow 0$$

be the Auslander-Reiten sequence ending at Y . Consider the induced sequence

$$e^{(K)} : 0 \longrightarrow \tau_{R^{(K)}} Y^{(K)} \longrightarrow E^{(K)} \longrightarrow Y^{(K)} \longrightarrow 0.$$

It is well known (see [1, Chapter V, Proposition 2.1], [12, proof of Theorem 11.27]) that e is represented by an element of the two-sided socle S of the $\text{End}_R(\tau_R Y)$ - $\text{End}_R(Y)$ -bimodule $\text{Ext}_R^1(Y, \tau_R Y)$. It follows from Corollary 3.5 (b) that $S^{(K)}$ is

the two-sided socle S' of the $\text{End}_{R^{(K)}}(\tau_{R^{(K)}}Y^{(K)})$ - $\text{End}_{R^{(K)}}(Y^{(K)})$ -bimodule

$$\text{Ext}_{R^{(K)}}^1(Y^{(K)}, \tau_{R^{(K)}}Y^{(K)})$$

and the sequence $e^{(K)}$ is represented by an element $\varepsilon \in S'$.

Fix a decomposition $Y^{(K)} \cong \bigoplus_{i=1}^r Z_i$ into indecomposable $R^{(K)}$ -modules Z_i . It follows from Lemmata 2.1 and 2.5 that the modules Z_i are not projective. This decomposition induces a direct sum decomposition

$$\text{Ext}_{R^{(K)}}^1(Y^{(K)}, \tau_{R^{(K)}}Y^{(K)}) = \bigoplus_{i,j=1}^n \text{Ext}_{R^{(K)}}^1(Z_i, \tau_{R^{(K)}}Z_j).$$

Observe first that $S' = \bigoplus_{i,j=1}^n S_{ij}$, where $S_{ij} = S' \cap \text{Ext}_{R^{(K)}}^1(Z_i, \tau_{R^{(K)}}Z_j)$ for $i, j = 1, \dots, n$. This follows from the fact that $p'_j s p_i \in \text{Ext}_{R^{(K)}}^1(Z_i, \tau_{R^{(K)}}Z_j) \cap S'$ for any $s \in S'$, where p_i and p'_j are the composition of the canonical maps $\bigoplus_{l=1}^n Z_l \rightarrow Z_i \rightarrow \bigoplus_{l=1}^n \tau_{R^{(K)}}Z_l$ and $\bigoplus_{l=1}^n \tau_{R^{(K)}}Z_l \rightarrow \tau_{R^{(K)}}Z_j \rightarrow \bigoplus_{l=1}^n \tau_{R^{(K)}}Z_l$ respectively for $i, j = 1, \dots, n$.

Next assume that $Z_i \not\cong Z_j$. We show that $S_{ij} = 0$. Take an arbitrary element η_{ij} of S_{ij} representing an exact sequence

$$0 \longrightarrow \tau_{R^{(K)}}Z_j \longrightarrow X \longrightarrow Z_i \longrightarrow 0.$$

If this sequence does not split, then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_{R^{(K)}}Z_j & \longrightarrow & F_j & \longrightarrow & Z_j & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow g & & \\ 0 & \longrightarrow & \tau_{R^{(K)}}Z_j & \longrightarrow & X & \longrightarrow & Z_i & \longrightarrow & 0 \end{array}$$

where the first row is an Auslander-Reiten sequence. Since the homomorphism g is not an isomorphism, it follows that η_{ij} is not annihilated by $\text{rad End}_{R^{(K)}}(Y^{(K)})$ from the right and thus $\eta_{ij} \notin S_{ij}$.

Moreover it follows from [1, Proposition 2.2] that if $Z_i \cong Z_j$, then S_{ij} coincides with the two-sided socle \overline{S}_{ij} of the $\text{End}_{R^{(K)}}(\tau_{R^{(K)}}Z_j)$ - $\text{End}_{R^{(K)}}(Z_i)$ -bimodule $\text{Ext}_{R^{(K)}}^1(Z_i, \tau_{R^{(K)}}Z_j)$. This bimodule is simple as a left $\text{End}_{R^{(K)}}(\tau_{R^{(K)}}Z_j)$ -module and as a right $\text{End}_{R^{(K)}}(Z_i)$ -module and its nonzero elements represent almost split sequences

$$0 \longrightarrow \tau_{R^{(K)}}Z_j \longrightarrow F \longrightarrow Z_i \longrightarrow 0.$$

We claim that for any $\eta = [\eta_{ij}] \in S'$ there exist invertible elements $f \in \text{End}_{R^{(K)}}(Y^{(K)})$ and $g \in \text{End}_{R^{(K)}}(\tau_{R^{(K)}}Y^{(K)})$ such that if $g\eta f = [\overline{\eta}_{ij}]$, then $\overline{\eta}_{ij} = 0$ for $i \neq j$.

Since, as we have shown, $S_{ij} = 0$ if $Z_i \not\cong Z_j$, without loss of generality we may assume that $Z_1 \cong \dots \cong Z_n$. Then our claim follows from Lemma 3.7.

We have just proved that the sequence

$$0 \longrightarrow \tau_{R^{(K)}}Y^{(K)} \longrightarrow E^{(K)} \longrightarrow Y^{(K)} \longrightarrow 0$$

is isomorphic to the direct sum of exact sequences

$$0 \longrightarrow \tau_{R^{(K)}}Z_i \longrightarrow F_i \longrightarrow Z_i \longrightarrow 0$$

represented by elements $\eta_{ii} \in S_{ii}$ for $i = 1, \dots, n$.

Assume that some η_{ii} is zero. It means that Z_i is a direct summand of F_i and since $E^{(K)} \cong \bigoplus_{i=1}^r F_i$ it follows by Lemma 2.5(a) that Y is a direct summand of E , a contradiction.

It follows that all elements η_{ii} are nonzero and they correspond to almost split sequences in $\text{mod}(R^{(K)})$ (see [12, Theorem 11.27]). \square

4. APPLICATIONS TO IRREDUCIBLE MORPHISMS
AND AUSLANDER-REITEN SEQUENCES

Lemma 4.1. *Assume that the field extension $k \subseteq K$ is MacLane separable. Assume that X is an R -module and $X^{(K)} \cong \bigoplus_{i=1}^r X_i$ is a decomposition into indecomposable direct summands. If there exists an irreducible morphism $X_i \rightarrow Z$ (resp. $Z \rightarrow X_i$) in $\text{mod}(R^{(K)})$ and Z is indecomposable, then there exist an indecomposable R -module Y and an irreducible morphism $X \rightarrow Y$ (resp. $Y \rightarrow X$) such that Z is a direct summand of $Y^{(K)}$.*

Proof. If there exists an irreducible morphism $X_i \rightarrow Z$ in $\text{mod}(R^{(K)})$, then X_i is not simple injective and hence by Corollaries 2.5 and 3.5(b) X is neither. Moreover it follows from Theorem 3.8 that Z is a direct summand of $E^{(K)}$, where E is a codomain of the minimal left almost split morphism $X \rightarrow E$ in $\text{mod}(R)$. We take for Y an indecomposable direct summand of E such that Z is a direct summand of $Y^{(K)}$. In case of an irreducible morphism $Z \rightarrow X_i$ the proof is analogous. \square

Proposition 4.2. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let X, Y be indecomposable R -modules and let $X^{(K)} = \bigoplus_{i=1}^r X_i^{n_i}, Y^{(K)} = \bigoplus_{j=1}^s Y_j^{m_j}$, where the modules X_1, \dots, X_r (resp. Y_1, \dots, Y_s) are indecomposable and pairwise nonisomorphic.*

- (1) *The following conditions are equivalent.*
 - (a) *There is an irreducible morphism $X \rightarrow Y$ in $\text{mod}(R)$.*
 - (b) *For any $1 \leq i \leq r$ there exist $1 \leq j \leq s$ and an irreducible morphism $X_i \rightarrow Y_j$ in $\text{mod}(R^{(K)})$.*
 - (c) *For any $1 \leq j \leq s$ there exist $1 \leq i \leq r$ and an irreducible morphism $X_i \rightarrow Y_j$ in $\text{mod}(R^{(K)})$.*
 - (d) *For some $1 \leq j \leq s$ and $1 \leq i \leq r$ there is an irreducible morphism $X_i \rightarrow Y_j$ in $\text{mod}(R^{(K)})$.*
- (2) *If the above conditions (a)–(d) hold and the arrow $X \rightarrow Y$ in Γ_R has valuation (a, b) , then*

$$n_i a = \sum_{j=1}^s m_j a_{ij}$$

for any $i = 1, \dots, r$ and

$$m_j b = \sum_{i=1}^r n_i b_{ij}$$

for any $j = 1, \dots, s$, where (a_{ij}, b_{ij}) is the valuation of the arrow $X_i \rightarrow Y_j$ in $\Gamma_{R^{(K)}}$ if such an arrow exists and $a_{ij} = b_{ij} = 0$ otherwise.

Proof. Assume that there is an irreducible map $X \rightarrow Y$ and the corresponding arrow in Γ_R has valuation (a, b) . Consider the right almost split morphism

$$X^a \oplus X' \rightarrow Y$$

where X' has no direct summand isomorphic to X . By Theorem 3.8 the induced homomorphism

$$(X^{(K)})^a \oplus X'^{(K)} \rightarrow Y^{(K)}$$

splits into a direct sum of minimal right almost split morphisms $Z_j \rightarrow Y_j$ for $j = 1, \dots, s$ and since each X_i is a direct summand of some Z_j the condition (b) in (1) follows. We have just shown the implication (a) \Rightarrow (b). Moreover, calculating multiplicities of X_i and Y_j in the direct sum decompositions of $(X^{(K)})^a \oplus X'^{(K)}$ and $Y^{(K)}$ we easily obtain the first equality in the statement (2). Observe that the module $X'^{(K)}$ has no direct summand isomorphic to X_i by Lemma 2.5.

The implication (d) \Rightarrow (a) follows from Lemma 4.1 and 2.5(a). The remaining implications in (1) as well as the second equality in (2) can be proved analogously. □

Remark 4.3. The result above as well as Theorem 3.8 has flavour similar to the results of I. Reiten and Ch. Riedtmann [8, 4.1] concerning relationships between irreducible maps and almost split sequences over an Artin algebra Λ and a skew group algebra $\Lambda * G$. It would be interesting to work out a universal framework containing these facts.

Proposition 4.4. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let \mathcal{C} be a connected component of Γ_R . There exists a family $\{\mathcal{D}_s\}_{s \in S}$ of connected components of $\Gamma_{R^{(K)}}$ having the following properties.*

(1) *If X belongs to \mathcal{C} and $X^{(K)}$ decomposes into a direct sum of indecomposable $R^{(K)}$ -modules X_1, \dots, X_r , then for each $s \in S$ there exists $1 \leq i \leq r$ such that X_i belongs to \mathcal{D}_s . In particular the family $\{\mathcal{D}_s\}_{s \in S}$ is finite.*

(2) *For any $s \in S$ each module in \mathcal{D}_s is a direct summand of $Y^{(K)}$ for some Y in \mathcal{C} .*

Proof. For $\{\mathcal{D}_s\}_{s \in S}$ we take the family of those connected components of $\Gamma_{R^{(K)}}$ which contain at least one direct summand of the module $Y^{(K)}$ for a module Y in \mathcal{C} .

(1) Let Y be a module in \mathcal{C} and Y' an indecomposable direct summand of $Y^{(K)}$ lying in \mathcal{D}_s . We shall prove that X_i belongs to \mathcal{D}_s for some $1 \leq i \leq r$. We proceed by induction on the distance $\text{dist}_{\mathcal{C}}(X, Y)$ of X from Y in \mathcal{C} .

If this distance is zero, that is, $X \cong Y$, there is nothing to prove. Assume the distance is positive and there is an irreducible morphism $X \rightarrow Z$ or $Z \rightarrow X$ in $\text{mod}(R)$ with $\text{dist}_{\mathcal{C}}(Z, Y) < \text{dist}_{\mathcal{C}}(X, Y)$. Consider the case $X \rightarrow Z$; the remaining one is analogous. Let $Z^{(K)} = \bigoplus_{j=1}^s Z_j$ with Z_j indecomposable. By the induction hypothesis Z_j belongs to \mathcal{D}_s for some $1 \leq j \leq s$. By Proposition 4.2 there exists an irreducible morphism $X_i \rightarrow Z_j$ for some i and hence X_i belongs to \mathcal{D}_s as we required.

The assertion (2) follows directly from Lemma 4.1. □

By Lemmata 2.1, 2.5 and Corollaries 2.4, 3.6 an indecomposable R -module X belongs to a τ_R -orbit of a projective (resp. injective) R -module if and only if each indecomposable direct summand of $X^{(K)}$ belongs to a $\tau_{R^{(K)}}$ -orbit of a projective (resp. injective) $R^{(K)}$ -module. Hence the Proposition 4.4 is also valid for a stable (resp. semi-stable) component \mathcal{C} of Γ_R .

Proposition 4.5. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let \mathcal{C} be a stable (resp. semi-stable) component of Γ_R . There exists a family $\{\mathcal{D}_s\}_{s \in S}$ of stable (resp. semi-stable) components in $\Gamma_{R^{(K)}}$ having the following properties.*

(1) If X belongs to \mathcal{C} and $X^{(K)}$ decomposes into a direct sum of indecomposable $R^{(K)}$ -modules X_1, \dots, X_r , then for each $s \in S$ there exists $1 \leq i \leq r$ such that X_i belongs to \mathcal{D}_s . In particular the family $\{\mathcal{D}_s\}_{s \in S}$ is finite.

(2) For any $s \in S$ each module in \mathcal{D}_s is a direct summand of $Y^{(K)}$ for some Y in \mathcal{C} .

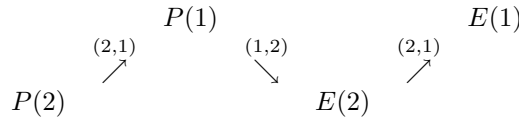
If \mathcal{C} and $\{\mathcal{D}_s\}_{s \in S}$ are as Proposition 4.4 or 4.5, we denote by $\mathcal{C}^{(K)}$ the subquiver $\bigcup_{s \in S} \mathcal{D}_s$ of $\Gamma_{R^{(K)}}$.

The following result is due to Jensen and Lenzing [4] (see also [5, Proposition 12.38, Corollary 12.39]).

Corollary 4.6. *Assume that the field extension $k \subseteq K$ is MacLane separable. If R is of finite representation type, then the algebra $R^{(K)}$ is of finite representation type and each indecomposable $R^{(K)}$ -module is a direct summand of a module $X^{(K)}$ for some indecomposable R -module X .*

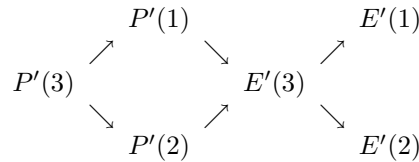
Proof. The quiver Γ_R has only finite components. It follows by Lemma 2.1 and Proposition 4.4 that each connected component of $\Gamma_{R^{(K)}}$ containing a projective $R^{(K)}$ -module is finite. Thus by well-known arguments (see e.g. [1, Chapter IV, Theorem 1.4]) we see that the algebra $R^{(K)}$ is of finite representation type and each component of $\Gamma_{R^{(K)}}$ contains a projective module. The remaining statement of the corollary follows again by Proposition 4.4. \square

Example 4.7. Let $R = \begin{pmatrix} \mathbb{C} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. The Auslander-Reiten quiver Γ_R has the following form:



The valuations of the arrows are marked at the above figure. By $P(i)$ (resp. $E(i)$) we denote the indecomposable projective (resp. injective) R -module corresponding to the idempotent matrix e_i having 1 at the (i, i) -th place and zeros elsewhere.

The algebra $R^{(\mathbb{C})}$ is isomorphic to $\begin{pmatrix} \mathbb{C} & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}$ and the Auslander-Reiten quiver $\Gamma_{R^{(\mathbb{C})}}$ has the following form:



Here $P'(i)$ (resp. $E'(i)$) denotes the i -th indecomposable projective (resp. injective) $R^{(\mathbb{C})}$ -module.

Proposition 4.8. *Assume that the field extension $k \subseteq K$ is MacLane separable. If \mathcal{C} is a connected component (resp. stable, semi-stable component) of Γ_R in which all arrows have trivial valuations, X is a module in \mathcal{C} such that $X^{(K)}$ is indecomposable, then for each Y in \mathcal{C} the induced module $Y^{(K)}$ is indecomposable. In particular $\mathcal{C}^{(K)} \cong \mathcal{C}$ as translation quivers and each module in $\mathcal{C}^{(K)}$ is defined over k .*

Proof. We proceed by induction on the distance of Y from X in \mathcal{C} . If the distance is zero, then $X \cong Y$ and there is nothing to prove. Let Z be a module in \mathcal{C} such that $\text{dist}_{\mathcal{C}}(X, Z) < \text{dist}_{\mathcal{C}}(X, Y)$ and there is an irreducible morphism $Y \rightarrow Z$ or $Z \rightarrow Y$ in $\text{mod}(R)$. Consider the case of $Y \rightarrow Z$; the remaining one is dual. Then by the induction hypothesis the module $Z^{(K)}$ is indecomposable. Assume that $Y^{(K)} \cong \bigoplus_{i=1}^r Y_i^{m_i}$ where the modules Y_1, \dots, Y_r are indecomposable and pairwise nonisomorphic. It follows from Proposition 4.2 that for any $j = 1, \dots, r$ there is an arrow $Y_j \rightarrow Z^{(K)}$ in $\Gamma_{R^{(K)}}$ and if (a_j, b_j) is the valuation of this arrow, then $a = \sum_{j=1}^r m_j a_j$, where (a, b) is the valuation of the arrow $Y \rightarrow Z$ in Γ_R . Since $a = 1$ by our assumption, it follows that $r = 1$ and $m_1 = 1$ and $Y^{(K)}$ is indecomposable. \square

Example 4.9. Let $k \subseteq K$ be the field extension from Remark 3.4 and let $R = k[x]/((x^2 + t)^m)$ for some natural m . The Auslander-Reiten quiver of R has the form

$$k[x]/((x^2 + t)^m) \xleftrightarrow{\quad} k[x]/((x^2 + t)^{m-1}) \xleftrightarrow{\quad} \dots \xleftrightarrow{\quad} k[x]/(x^2 + t).$$

The algebra $R^{(K)}$ is isomorphic to $K[x]/(x + s)^{2m}$, where s is an element of K such that $s^2 = t$. Hence the Auslander-Reiten quiver of $R^{(K)}$ has the following form:

$$K[x]/((x + s)^{2m}) \xleftrightarrow{\quad} K[x]/((x + s)^{2m-1}) \xleftrightarrow{\quad} \dots \xleftrightarrow{\quad} K[x]/(x + s).$$

Observe that not each module in this quiver is defined over k .

Lemma 4.10. *Assume that the field extension $k \subseteq K$ is MacLane separable. Let A be the path algebra of a connected quiver Q with coefficients in K and let \mathcal{D} be a connected component in Γ_A . If a module X in \mathcal{D} is defined over k , then all modules in \mathcal{D} are defined over k .*

Proof. It is easy to check that all preprojective and preinjective modules are defined over the prime subfield of K .

Assume that \mathcal{D} is regular. Let $X \cong Y^{(K)}$ be an indecomposable module in \mathcal{D} and let \mathcal{C} be the connected component of Γ_R containing Y . It is clear that \mathcal{C} is a regular component in the Auslander-Reiten quiver of kQ and by [1, Theorem IV.4.15] either \mathcal{D} is a stable tube or it is isomorphic to $\mathbb{Z}A_\infty$ (see [1, Chapter VII.4] for the notation). Then the statement follows from Proposition 4.8. \square

Recall (see e.g. [10]) that an R -module is exceptional if it is indecomposable and has no selfextensions.

Corollary 4.11. *Let A be the path algebra of a quiver Q with coefficients in an algebraically closed field K and let \mathcal{D} be a component in Γ_A . If \mathcal{D} contains an exceptional module, then all modules in \mathcal{D} are defined over the prime subfield of K .*

Proof. Without loss of generality we may assume that \mathcal{D} is regular. By results of Ringel [10], see also [11], we know that an exceptional module is defined over the prime subfield F of K . Since prime fields are perfect, the extension $F \subseteq K$ is MacLane separable and our claim follows by Lemma 4.10. \square

We finish with a proposition describing a class of stable and semi-stable components of $\Gamma_{R^{(K)}}$ having the similar property as connected components of the Auslander-Reiten quiver of a hereditary algebra.

Proposition 4.12. *Assume that the field extension $k \subseteq K$ is MacLane separable and assume that \mathcal{D} is an infinite stable (resp. semi-stable) component in $\Gamma_{R^{(K)}}$ containing a cycle of irreducible morphisms. If there is a module X in \mathcal{D} defined over k , then each module in \mathcal{D} is defined over k .*

Proof. Let $X = Y^{(K)}$ for an indecomposable R -module Y and consider the stable (resp. semi-stable) component \mathcal{C} of Γ_R containing Y . Then by Proposition 4.5 \mathcal{D} is a connected component of $\mathcal{C}^{(K)}$. Next we note that there is a cycle of irreducible morphisms in \mathcal{C} . Indeed, let

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots \rightarrow X_r \rightarrow X_1$$

be a cycle in \mathcal{D} and X_i is direct summand of $Y_i^{(K)}$ for an indecomposable R -module Y_i in \mathcal{C} . It follows by Lemma 4.1 that then there is a cycle

$$Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow \dots \rightarrow Y_r \rightarrow Y_1$$

in \mathcal{C} .

Now by results of S. Liu (see [7], [6]) each arrow in \mathcal{C} has trivial valuation and hence by Proposition 4.8 each module in \mathcal{D} is defined over k . \square

We finish the paper with the remark about the case of an algebraic separable extension $k \subseteq K$.

Proposition 4.13. *Assume that the extension $k \subseteq K$ is separable algebraic. Then any $R^{(K)}$ -module is a direct summand of a module defined over k . Any connected component of the Auslander-Reiten quiver $\Gamma_{R^{(K)}}$ is a component of $\mathcal{C}^{(K)}$ for a connected component \mathcal{C} of Γ_R .*

Proof. The proof follows [4]. Take an $R^{(K)}$ -module Y . We shall prove that Y is a direct summand of a module defined over k . The module Y is defined over a finite extension of k ; hence we can assume that the extension $k \subseteq K$ is finite separable. Then the K - K -bimodule $K \otimes_k K$ is semisimple and the trivial K - K -bimodule K is a direct summand of $K \otimes_k K$. The module Y has a natural structure of an R -module. The module $Y \otimes_k K \cong Y \otimes_K K \otimes_k K$ which is defined over k contains Y as a direct summand.

The remaining statement follows now from Proposition 4.4. \square

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