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NONCOMPLEX SMOOTH 4-MANIFOLDS WITH GENUS-2 LEFSCHETZ FIBRATIONS

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Abstract. We construct noncomplex smooth 4-manifolds which admit genus-2 Lefschetz fibrations over S^2. The fibrations are necessarily hyperelliptic, and the resulting 4-manifolds are not even homotopy equivalent to complex surfaces. Furthermore, these examples show that fiber sums of holomorphic Lefschetz fibrations do not necessarily admit complex structures.

In this paper we will prove the following theorem.

Theorem 1.1. *There are infinitely many (pairwise nonhomeomorphic) 4-manifolds which admit genus-2 Lefschetz fibrations but do not carry complex structure with either orientation.*

Matsumoto [6] showed that $S^2 \times T^2 \# 4\overline{\mathbb{C}P^2}$ admits a genus-2 Lefschetz fibration over S^2 with global monodromy $(\beta_1, ..., \beta_4)^2$, where $\beta_1, ..., \beta_4$ are the curves indicated by Figure 1. (For definitions and details regarding Lefschetz fibrations see [6], [5].)

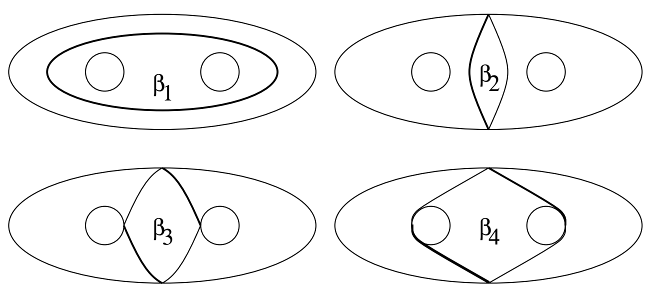

Figure 1.

Let B_n denote the smooth 4-manifold which admits a genus-2 Lefschetz fibration over S^2 with global monodromy

$$((\beta_1, ..., \beta_4)^2, (h^n(\beta_1), ..., h^n(\beta_4))^2)$$

where $h = D(a_2)$ is a positive Dehn twist about the curve a_2 indicated in Figure 2.

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Theorem 1.2. *For the 4-manifold B_n given above we have $\pi_1(B_n) = \mathbb{Z} \oplus \mathbb{Z}_n$.*

Proof. Standard theory of Lefschetz fibrations gives that

$$\pi_1(B_n) = \pi_1(\Sigma_2)/\langle \beta_1, ..., \beta_4, h^n(\beta_1), ..., h^n(\beta_4)\rangle.$$

Let $\{a_1, b_1, a_2, b_2\}$ be the standard generators for $\pi_1(\Sigma_2)$ (Figure 2).

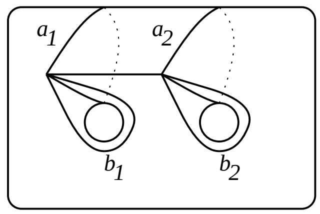

FIGURE 2.

Then we observe that
$\beta_1 = b_1 b_2$,
$\beta_2 = a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1}$,
$\beta_3 = b_2 a_2 b_2^{-1} a_1$,
$\beta_4 = b_2 a_2 a_1 b_1$,
$h^n(\beta_1) = b_1 b_2 a_2^n$,
$h^n(\beta_2) = \beta_2$,
$h^n(\beta_3) = \beta_3$,
$h^n(\beta_4) = b_2 a_2^{n+1} a_1 b_1$.
Hence

$$\begin{aligned}\pi_1(B_n) &= \langle a_1, b_1, a_2, b_2 \mid b_1 b_2,\ [a_1, b_1],\ b_2 a_2 b_2^{-1} a_1,\ b_2 a_2 a_1 b_1,\ b_1 b_2 a_2^n,\ b_2 a_2^{n+1} a_1 b_1\rangle \\ &= \langle a_2, b_2 \mid [a_2, b_2],\ a_2^n\rangle \\ &= \mathbb{Z} \oplus \mathbb{Z}_n,\end{aligned}$$

and this concludes the proof. □

The above definition of B_n provides a handlebody decomposition for it [5] and shows, in particular, that the Euler characteristic $\chi(B_n)$ is equal to 12. Since B_n is the fiber sum of two copies of $S^2 \times T^2 \# 4\overline{\mathbb{C}P^2}$, we get that the signature $\sigma(B_n) = -8$. Consequently, $b_2(B_n) = 12$ and $b_2^+(B_n) = 2$, $b_2^-(B_n) = 10$. Let M_n deonote the n-fold cover of B_n with $\pi_1(M_n) \cong \mathbb{Z}$. Easy computation shows that $b_2^+(M_n) = 2n$ and $b_2^-(M_n) = 10n$.

Theorem 1.3. *B_n does not admit a complex structure.*

Proof. Assume that B_n admits a complex structure and let M_n' denote the minimal model of M_n. By the Enriques-Kodaira classification of complex surfaces [1] (since $b_1(M_n') = 1$), M_n' is either a surface of class VII (in which case $b_2^+(M_n') = 0$), a secondary Kodaira surface (in which case $b_2(M_n') = 0$) or a (minimal) properly elliptic surface. Since $b_2^+(M_n') = b_2^+(M_n) = 2n$, the first two possibilities are ruled out.

Suppose now that M_n' admits an elliptic fibration over a Riemann surface. If the Euler characteristic of M_n' is 0, then (following from the fact that $b_1(M_n') = b_3(M_n') = 1$) we get that $b_2(M_n') = 0$, which leads to the above contradiction. Suppose finally that M_n' is a minimal elliptic surface with positive Euler characteristic. Since $b_1(M_n') = 1$, it can only be fibered over S^2 (see for example [2]). In that case (according to [4], for example) its fundamental group is

$$\pi_1(M_n') = \langle x_1, \dots x_k \mid x_i^{p_i} = 1,\ i = 1, \dots, k;\ x_1 \cdots x_k = 1 \rangle.$$

This cannot be isomorphic to $\mathbb{Z}$, since if $\pi_1(M_n) \cong \mathbb{Z} = \langle a \rangle$, then $x_1 = a^{m_1}$ for some $m_1 \in \mathbb{Z}$, so a has finite order, which is a contradiction. Consequently the assumption that B_n is complex leads us to a contradiction; hence the theorem is proved. □

Remark 1.4. The above proof, in fact, shows that B_n is not even homotopy equivalent to a complex surface — our arguments used only homotopic invariants (the fundamental group, b_2 and b_2^+) of the 4-manifold B_n. Note that the same idea shows that $\overline{B}_n$ (the manifold B_n with the opposite orientation) carries no complex structure: The arguments involving the fundamental group, b_2 and the Euler characteristic only, apply without change. Since $b_2^+(\overline{M}_n) = b_2^-(M_n) = 10n \neq 0$, the arguments using that fact that $b_2^+ \neq 0$ apply as well.

Proof of Theorem 1.1. By the definition of the 4-manifolds B_n we get infinitely many manifolds admitting genus-2 (consequently hyperelliptic) Lefschetz fibrations which are (by Theorem 1.2) nonhomeomorphic. As Theorem 1.3 and the above remark show, the manifolds B_n do not carry complex structures with either orientation; hence the proof of the Theorem 1.1 is complete. □

Remark 1.5. We would like to point out that similar examples have been found by Fintushel and Stern [3] — they used Seiberg-Witten theory to prove that their (simply connected) Lefschetz fibrations are noncomplex.

Note that B_n is given as the fiber sum of two copies of $S^2 \times T^2 \# 4\overline{\mathbb{C}P^2}$, hence provides an example of the phenomenon that the fiber sum of holomorphic Lefschetz fibrations is not necessarily complex.

Acknowledgement

Examples of genus-2 Lefschetz fibrations with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ were also constructed (as fiber sums) independently by Ivan Smith [7].

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