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NEVANLINNA FUNCTIONS AS QUOTIENTS

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Abstract. Let f be a holomorphic function in the unit ball. Then f is a Nevanlinna function if and only if there exist Smirnov functions f_+, f_- such that $f = f_+/f_-$ and f_- has no zeros in the ball.

Let $B = B_n$ be the open unit ball in $\mathbb{C}^n$ and $S = \partial B$ be the unit sphere. If $n = 1$, then $\mathbb{D} = B_1$ is the open unit disc in $\mathbb{C}$.

The Nevanlinna class $N(B)$ is the set of all holomorphic functions f on B such that

$$\sup_{0<r<1} \int_S \log^+ |f_r|\, d\sigma < +\infty,$$

where σ is the normalized Lebesgue measure on S, $f_r(\zeta) = f(r\zeta)$, and $x^+ = \max(x, 0)$.

The Smirnov class $N^+(B)$ is the space of all functions $f \in N(B)$ such that the family $\{\log^+ |f_r|\}_{0<r<1}$ is uniformly integrable on S. Namely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_\Delta \log^+ |f_r|\, d\sigma < \varepsilon$$

for all $0 < r < 1$ and all sets $\Delta \subset S$ such that $\sigma(\Delta) < \delta$.

Let f be a holomorphic function on the disc. The classical factorization theorem implies that $f \in N(\mathbb{D})$ if and only if $f = f_+/f_-$, where f_+ and f_- are holomorphic and bounded by 1 on $\mathbb{D}$ and f_- has no zeros on $\mathbb{D}$. In particular, the functions f and f_+ have the same zero sets.

Now, assume $n \geq 2$. Then there are many ways to say that the factorization theorem does not hold in the ball B_n. For example, there exist many functions $f \in N(B)$, $f \not\equiv 0$, with the following property: If $f_+ \in H^\infty(B)$ satisfies f_+/f is holomorphic, then $f_+ \equiv 0$. Hence, the above characterization does not hold for $n \geq 2$. Moreover, the zero sets of the Hardy classes $H^p(B)$ are all different (see [5]). One more negative result in this direction is the following nonfactorization theorem: The set $\{gh : g, h \in H^2(B)\}$ is a set of first category in $H^1(B)$ (see [3]).

However, in the present note we obtain the following description of the Nevanlinna class in the ball.

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Theorem 1. *Let f be a holomorphic function on B_n, $n \geq 2$. Then $f \in N(B)$ if and only if there exist $f_+, f_- \in N^+(B)$ such that $f = f_+/f_-$ and $f_-(z) \neq 0$ for all $z \in B$.*

Proof. If $f_+, f_- \in N^+(B)$ and f_+/f_- is a holomorphic function, then f_+/f_- is a Nevanlinna function since

$$\sup_{\frac{1}{2} \leq r < 1} \int_S |\log |(f_-)_r|| \, d\sigma < \infty.$$

So assume $f \in N(B)$. The Henkin-Skoda theorem provides a function $f_0 \in N^+(B)$ which has the same zeros (see [4] and [6]). More precisely, $f = f_0 F$, where $F \in H(B)$ and $F(z) \neq 0$ for all $z \in B$. Without loss of generality $F(0) \in \mathbb{R}$.

Observe that $F \in N(B)$. Therefore $u = \log |F|$ is a pluriharmonic function and $\sup_{0<r<1} \int_S |u_r| d\sigma < \infty$. Hence, there exists a real measure μ on the sphere such that u is the Poisson integral $P[\mu]$. Let $\mu = \mu_s + \mu_a$ be the Lebesgue decomposition of μ (here $\mu_s \perp \sigma$ and $\mu_a \ll \sigma$). Also, let $\mu_s = \mu_+ - \mu_-$, $\mu_\pm \geq 0$, be the Jordan decomposition of μ_s. We have $\mu_+ \ll \mu_s$; thus μ_+ is the singular part of a measure ν such that the Poisson integral $P[\nu]$ is a pluriharmonic function in the ball ([2], Corollary 2.7). In other words, there exists a function $g \in L^1(\sigma)$ such that $P[g\sigma - \mu_+]$ is pluriharmonic. Denote by F_- the holomorphic function such that $\mathrm{Re}\, F_- = P[g\sigma - \mu_+]$ and $F_-(0) \in \mathbb{R}$. Respectively, the Poisson integral of the measure $\mu_a + g\sigma - \mu_- = \mu - (\mu_+ - g\sigma)$ is a pluriharmonic function. So let F_+ be the holomorphic function such that $\mathrm{Re}\, F_+ = P[\mu_a + g\sigma - \mu_-]$ and $F_+(0) \in \mathbb{R}$.

Clearly, the definitions of F_- and F_+ yield the identity $F = \exp F_+ \exp(-F_-)$. Put $f_- = \exp F_-$. Then $\log |(f_-)_r| = P_r[g\sigma - \mu_+] \leq P_r[g\sigma]$ since μ_+ is a nonnegative measure. So $\log^+ |(f_-)_r| \leq P_r[g^+\sigma]$, hence, the family $\left\{\log^+ |(f_-)_r|\right\}_{0<r<1}$ is uniformly integrable. In other words $f_- \in N^+(B)$.

Analogously $\exp F_+ \in N^+(B)$, and we put $f_+ = f_0 \exp F_+$. □

Observe that even a nonvanishing Nevanlinna function is not necessarily a quotient f_+/f_-, where $f_\pm \in H^\infty(B)$ and f_- has no zeros in the ball. Moreover, one has the following negative example.

Proposition 2. *There exists $f \in N(B)$ such that $f(z) \neq 0$ for all $z \in B$ and f cannot be represented as f_+/f_-, where $f_+ \in N^+(B)$, $f_- \in H^\infty(B)$ and f_- has no zeros in the ball.*

Proof. Fix a point $\zeta \in S$. Let m be the normalized Lebesgue measure on the circle $\mathbb{T}_\zeta = \{\lambda\zeta : \lambda \in \mathbb{T}\} \subset S$. Choose a strictly positive lower semicontinuous function $g \in L^1(\sigma)$ such that

$$\sup_{0<r<1} \int_S P_r[m] g \, d\sigma = +\infty.$$

Since g is lower semicontinuous, there exists a singular positive measure μ on S such that $P[g\sigma - \mu] = \mathrm{Re}\, F$ for certain $F \in H(B)$ (see [1], Chapter 5, 4.2). Put $f = \exp(-F) \in N(B)$.

Now, suppose $f = f_+/f_-$, where $f_\pm$ are nonvanishing, $f_+ \in N^+(B)$, and $f_- \in H^\infty(B)$. Since $f_\pm$ are zero-free, there exist measures $\nu_\pm \in M(S)$ such that $-\log |f_\pm| = P[\nu_\pm]$. Without loss of generality, let $\log |f_-| \leq 0$, in other words $\nu_- \geq 0$. Let $(\nu_\pm)_s$ be the singular parts of $\nu_\pm$; then $\mu = (\nu_-)_s - (\nu_+)_s$. Recall that

$f_+ \in N^+(B)$, so $(\nu_+)_s \geq 0$. Thus $\nu_- \geq (\nu_-)_s \geq \mu$. Therefore $g\sigma \leq g\sigma + \nu_- - \mu$ and the Poisson integral $u = P[(g\sigma - \mu) + \nu_-]$ is pluriharmonic, hence

$$u(0) = \sup_{0<r<1} \int_S u_r \, dm \geq \sup_{0<r<1} \int_S P_r[g\sigma] \, dm = +\infty.$$

A contradiction. □

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