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SEMIGROUPS AND WEIGHTS FOR GROUP REPRESENTATIONS

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ABSTRACT. Let G be a finite group. Consider a pair $\chi = (\chi_+, \chi_-)$ of linear characters of subgroups P, P^- of G with χ_+ and χ_- agreeing on $P \cap P^-$. Naturally associated with χ is a finite monoid M_χ. Semigroup representation theory then yields a representation θ of G. If θ is irreducible, we say that χ is a weight for θ. When the underlying field is the field of complex numbers, we obtain a formula for the character of θ in terms of χ_+ and χ_-. We go on to construct weights for some familiar group representations.

1. Introduction

A basic theme of representation theory is the construction of an irreducible representation from a linear character (degree 1 representation). In particular, this has been accomplished for finite Lie type groups in the defining characteristic by Curtis, Steinberg and Richen; cf. [5]. Reinterpreting these results, Alperin [1] came up with his famous weight conjectures for irreducible modular representations of arbitrary finite groups.

Motivated by semigroup representation theory [3, Chapter 5], we proceed in this paper in a completely different way. For a finite monoid M, the irreducible representations are indexed by the irreducible representations θ of maximal subgroups H of M. We have noted in [13] that θ gives rise to representations θ_+, θ_- (of the same degree) of subgroups P, P^- of G. We observe that even in the situation of irreducible modular representations of a Lie type group [5], a linear character θ of a Levi subgroup L lifts to linear characters θ_+ and θ_- of associated opposite parabolic subgroups P, P^-. In this situation there is an associated Lie type monoid [14]. In the ordinary representation theory of the symmetric group S_n (cf. [6, Section 28]), the irreducible representations are indexed by pairs of linear characters—one trivial and one alternating—of two Young subgroups of S_n. In this case too, there is a naturally associated monoid. Let e be the primitive idempotent associated with the Young diagram and let $J = S_n e S_n$. Then

$$M = S_n \cup J \cup \{0\}$$

is a monoid and all idempotents in J are conjugate, a special property shared by monoids of Lie type.

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Let G be a finite group with a pair of linear characters $\chi = (\chi_+, \chi_-)$ of subgroups P, P^-, with χ_+, χ_- agreeing on $P \cap P^-$. We then construct a monoid M_χ, which gives rise to a representation θ of G. When θ is irreducible, we say that χ is a weight for θ. Over $\mathbb{C}$, we determine the character of θ in terms of χ_+ and χ_-. After developing the general theory we construct weights for the Steinberg representation and the unipotent representations of $GL_n(\mathbb{F}_q)$.

2. Semigroups and weights

In this section we introduce our notion of a weight of a group representation, related to semigroup theory. We begin by reviewing some relevant semigroup theory. Let M be a finite monoid with unit group G. For $a, b \in M$, Green's relation $\mathcal{J}$ is defined as: $a \,\mathcal{J}\, b$ if $MaM = MbM$. For a $\mathcal{J}$-class J of M, we can form a semigroup

$$J^0 = J \cup \{0\}$$

where for $a, b \in J$,

$$a \circ b = \begin{cases} ab & \text{if } ab \in J, \\ 0 & \text{if } ab \notin J. \end{cases}$$

We can then also form the monoid

$$M(J) = G \cup J^0.$$

Our interest is in the situation when J^0 is not null, i.e. J has an idempotent e. In much of our work, beginning with [10], the following opposite "parabolic" subgroups have played a significant role:

$$\begin{aligned} P = P(e) &= \{x \in G | xe = exe\}, \\ P^- = P^-(e) &= \{x \in G | ex = exe\}. \end{aligned} \tag{1}$$

Let $H = H(e)$ denote the unit group of eMe. Then we have homomorphisms $\delta_+ \colon P \to H$, $\delta_- \colon P^- \to H$, agreeing on $P \cap P^-$, given by:

$$\delta_+(p) = pe, \quad \delta_-(q) = eq \quad \text{for } p \in P,\ q \in P^-. \tag{2}$$

The data (1), (2) does not uniquely determine the monoid $M(J)$. However, we have shown ([11, Theorem 1.1], [12, Theorem 1.3]) that if all the idempotents of J are conjugate, then $M(J)$ is uniquely determined by (1), (2). We then denote $M(J)$ by $M(G, P, P^-, H)$, with δ_+, δ_- understood to be part of the data.

Let F be an algebraically closed field. By a linear character of G, we will mean a representation of degree 1, i.e. a homomorphism into F^*. Let $\operatorname{Irr} G$ denote the set of irreducible representations of G over F. If $\theta \in \operatorname{Irr} G$, we let $\overline{\theta}$ denote the dual representation: $\overline{\theta}(g) = \theta(g^{-1})^t$. Let $M = M(J)$ and assume that the characteristic of F does not divide $|H|$. Let FM, FJ denote the contracted semigroup algebras of M and J, respectively. Thus the zero of J^0 is the zero of FJ. Clearly FJ is an ideal of FM. By semigroup representation theory ([3], [4, Chapter 5]),

$$\begin{aligned} FJ &\cong \bigoplus_{\theta \in \operatorname{Irr} H} \mathcal{A}_\theta, \\ FJ/\operatorname{rad} FJ &\cong \bigoplus_{\theta \in \operatorname{Irr} H} \mathcal{B}_\theta \end{aligned} \tag{3}$$

where

$$\mathcal{B}_\theta = \mathcal{A}_\theta / \operatorname{rad} \mathcal{A}_\theta$$

is a simple algebra. Here "rad" is the radical. Since $FJ/\operatorname{rad} FJ$ is an ideal of $FM/\operatorname{rad} FJ$, we have irreducible representations $\hat{\theta}\colon FM \to \mathcal{B}_\theta$, $\theta \in \operatorname{Irr} H$. These then restrict to representations $\tilde{\theta}$ of G, $\theta \in \operatorname{Irr} H$. We refer to [13] for details. Of particular importance to us is the situation when θ is a linear character. Then we have linear characters $\chi_+ = \theta \circ \delta_+$ and $\chi_- = \theta \circ \delta_-$ of P and P^-, that agree on $P \cap P^-$.

We now reverse the above analysis. Let P, P^- be subgroups of a finite group G. Let χ_+, χ_- be linear characters of P and P^- that agree on $P \cap P^-$. Let $\chi = (\chi_+, \chi_-)$. Now the subgroup H of F^* generated by $\chi_+(P)$ and $\chi_-(P^-)$ is a finite group of order not divisible by the characteristic of F. We can therefore form the monoid $M_\chi = M(J) = M(G, P, P^-, H)$; cf. [11, Theorem 1.1]. Let $\pi\colon H \to F^*$ denote the identity map. As in (3), let

$$\mathcal{A}_\chi = \mathcal{A}_\pi, \qquad \mathcal{B}_\chi = \mathcal{B}_\pi, \\ \hat{\mathcal{A}}_\chi = FG + \mathcal{A}_\chi. \tag{4}$$

Then $\hat{\mathcal{A}}_\chi$ is the algebra over F generated by G and an idempotent e, subject to the relations:

$$pe = \chi_+(p)\cdot e, \quad eq = \chi_-(q)\cdot e \quad \text{for } p \in P,\ q \in P^-, \\ ege = 0 \quad \text{if } g \in G,\ g \notin P^-P. \tag{5}$$

$\mathcal{A}_\chi$ is the span of GeG and is an ideal of $\hat{\mathcal{A}}_\chi$. It will be convenient for us to view χ also as a function with support P^-P:

$$\chi(qp) = \chi_-(q)\chi_+(p) \quad \text{if } p \in P,\ q \in P^-, \\ \chi(g) = 0 \quad \text{if } g \in G\backslash P^-P. \tag{6}$$

Let the right cosets of P^- and left cosets of P be, respectively,

$$P^-a_1, \dots, P^-a_m, \\ b_1P, \dots, b_nP. \tag{7}$$

Then $\mathcal{A}_\chi$ is a Munn algebra in the sense of [4, Section 5.2] with sandwich matrix,

$$\Delta = \Delta_\chi = (\chi(a_ib_j)). \tag{8}$$

We have the representation $\tilde{\pi}\colon G \to \mathcal{B}_\chi$. If $\tilde{\pi}$ is irreducible, then we say that $\chi = (\chi_+, \chi_-)$ is a *weight* (for $\tilde{\pi}$) and define

$$[\chi] = [\chi_+, \chi_-] = \tilde{\pi}. \tag{9}$$

Let $\theta\colon G \to GL(n, F)$ be an irreducible representation. Let $M_n(F)$ denote the algebra of all $n \times n$ matrices over F. Let ϵ be a primitive idempotent in $M_n(F)$. We may assume that $\epsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Let $g \in G$. Then

$$A = \theta(g) = \begin{bmatrix} a & b \\ C & D \end{bmatrix}, \qquad \epsilon A \epsilon = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

If $a \neq 0$, then A has an LU-decomposition:

$$A = LU, \qquad L = \begin{bmatrix} c & 0 \\ X & Y \end{bmatrix},\ U = \begin{bmatrix} d & Z \\ 0 & W \end{bmatrix}. \tag{10}$$

We will say that G has *LU-decomposition* with respect to ϵ if L, U can be chosen to be in $\theta(G)$ for all $g \in G$ with $\epsilon\theta(g)\epsilon \neq 0$.

Theorem 2.1. *Let $\theta\colon G \to GL(n,F)$ be an irreducible representation. Then*

(i) *θ has a weight if and only if G has LU-decomposition with respect to some primitive idempotent ϵ of $M_n(F)$.*
(ii) *$\theta = [\chi_+, \chi_-]$ with χ as in (9), if and only if for some primitive idempotent ϵ of $M_n(F)$, and all $p \in P$, $q \in P^-$, $g \in G \backslash P^-P$, $\theta(p)\epsilon = \chi_+(p)\epsilon$, $\epsilon\theta(q) = \chi_-(q)\epsilon$ and $\epsilon\theta(g)\epsilon = 0$.*

Proof. First we prove (ii). Suppose $\theta = [\chi_+, \chi_-]$. Then $\mathcal{B}_\chi \cong M_n(F)$. By (5), the conditions are satisfied with ϵ being the image of e. Conversely if the conditions are satisfied, then by (5), θ extends to a homomorphism from $\hat{\mathcal{A}}_\chi$ to $M_n(F)$ by sending e to ϵ. It follows that $\mathcal{B}_\chi \cong M_n(F)$ and that $\theta = [\chi]$.

We now prove (i). Suppose G has LU-decomposition with respect to a primitive idempotent ϵ of $M_n(F)$. Let

$$P = \{x \in G | \theta(x)\epsilon = \epsilon\theta(x)\epsilon\},$$
$$P^- = \{x \in G | \theta(x)\epsilon = \epsilon\theta(x) = \epsilon\theta(x)\epsilon\}.$$

Let $p \in P$. Since ϵ is primitive, $\theta(p)\epsilon = \chi_+(p)\epsilon$ for some $\chi_+(p) \in F^*$. Similarly for $q \in P^-$, $\epsilon\theta(q) = \chi_-(q)\epsilon$ for some $\chi_-(q) \in F^*$. Then clearly χ_+, χ_- are linear characters of P and P^-, agreeing on $P \cap P^-$. If $g \in G$, then by definition, $\epsilon\theta(g)\epsilon \neq 0$ implies that $g \in P^-P$. By (ii), $\chi = (\chi_+, \chi_-)$ is a weight for θ. Conversely, if $\theta = [\chi_+, \chi_-]$, then by (ii), G has LU-decomposition with respect to some primitive idempotent ϵ of $M_n(F)$.

Example 2.2. Let G be a finite Lie type group defined over $\mathbb{F}_q$. Let S denote the set of simple reflections. For $I \subseteq S$, let P_I, P_I^-, L_I denote the associated opposite parabolic subgroups and Levi subgroup, respectively. By [5], there is a 1-1 correspondence between the irreducible representations of G over $F = \overline{\mathbb{F}}_q$ and pairs (I, λ), where $I \subseteq S$, and λ is a linear character of L_I. If a representation θ of G corresponds to (I, λ), let $\chi_+ = \lambda \circ \delta_+$, $\chi_- = \lambda \circ \delta_-$, where $\delta_+\colon P_I \to L_I$, $\delta_-\colon P_I^- \to L_I$ are the natural homomorphisms. Then by [14] and Theorem 2.1, θ has weight (χ_+, χ_-). In this case (χ_-, χ_+) is also a weight, but in general $[\chi_-, \chi_+] \neq [\chi_+, \chi_-]$. This is because P_I need not be conjugate to P_I^-. Contrast this with Theorem 3.1.

Theorem 2.3. *Suppose $\chi = (\chi_+, \chi_-)$ is a weight. Then*

(i) *$\overline{\chi} = (\overline{\chi}_-, \overline{\chi}_+)$ is also a weight and $[\overline{\chi}] = \overline{[\chi]}$.*
(ii) *The degree of $[\chi]$ is equal to the rank of the matrix Δ_χ of (8).*

Proof. $\hat{\mathcal{A}}_\chi$ is given by the relations in (5). $\hat{\mathcal{A}}_{\overline{\chi}}$ is the algebra generated by G and an idempotent $\overline{e}$, subject to the relations:

$$q\overline{e} = \overline{\chi}_-(q)\overline{e}, \quad \overline{e}p = \overline{\chi}_+(p)\overline{e} \quad \text{for } p \in P,\ q \in P^-,$$
$$\overline{e}g\overline{e} = 0 \quad \text{for } g \in G,\ g \notin PP^-.$$

The map sending e to $\overline{e}$ and g to g^{-1}, $g \in G$, yields an anti-isomorphism between $\hat{\mathcal{A}}_\chi$ and $\hat{\mathcal{A}}_{\overline{\chi}}$. This yields a natural anti-isomorphism between $\mathcal{B}_\chi$ and $\mathcal{B}_{\overline{\chi}}$. It follows that $\overline{\chi}$ is a weight and that $[\overline{\chi}]$ is the dual of $[\chi]$.

By [3] or [4, Chapter 5], the degree of $[\chi]$ is equal to the rank of the sandwich matrix which is given in (8). This proves (ii). ∎

3. Complex representations

In this section we will let $F = \mathbb{C}$. If φ, ψ are characters of G, then the *intertwining number* is

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

If φ is a character of a subgroup P of G, then the *induced character* is

$$\varphi \uparrow G(g) = \frac{1}{|P|} \sum_{\substack{x \in G \\ xgx^{-1} \in P}} \varphi(xgx^{-1}).$$

Theorem 3.1. *Let $\chi = (\chi_+, \chi_-)$ be a weight. Then*

(i) (χ_-, χ_+) *is also a weight and* $[\chi] = [\chi_-, \chi_+]$.

(ii) *Let $[\chi]$ have degree m. Then with the notation* (6)*, the character ξ of $[\chi]$ is given by:*

$$\xi(g) = \frac{m}{|G|} \sum_{x \in G} \chi(xgx^{-1}).$$

Proof. (i) We have an anti-automorphism of $\mathbb{C}G$ given by:

$$(\Sigma \alpha_g g)^* = \Sigma \overline{\alpha}_g g^{-1}. \tag{11}$$

This anti-automorphism fixes the central idempotents and hence the blocks of $\mathbb{C}G$. By Theorem 2.1, G has LU-decomposition relative to a primitive idempotent ϵ in the block of $[\chi]$ such that

$$\begin{gathered} p\epsilon = \chi_+(p)\epsilon, \quad \epsilon q = \chi_-(q)\epsilon, \qquad p \in P,\ q \in P^-, \\ \epsilon g \epsilon = 0 \quad \text{if } g \in G \backslash P^- P. \end{gathered} \tag{12}$$

Then ϵ^* is also a primitive idempotent in the block of $[\chi]$ and

$$\begin{gathered} \epsilon^* p = \chi_+(p)\epsilon^*, \quad q\epsilon^* = \chi_-(q)\epsilon^*, \qquad p \in P,\ q \in P^-, \\ \epsilon^* g \epsilon^* = 0 \quad \text{if } g \notin PP^-. \end{gathered}$$

By Theorem 2.1, (χ_-, χ_+) is also a weight for $[\chi]$.

(ii) Now $\mathbb{C}G$ is a semisimple algebra with

$$\mathbb{C}G \cong \bigoplus_{\theta \in \operatorname{Irr} G} \mathcal{C}_\theta \tag{13}$$

where the simple algebra $\mathcal{C}_\theta$ is the block of θ. The identity element of $\mathcal{C}_\theta$ is:

$$\eta_\theta = \frac{1}{|G|} \sum_{x \in G} \overline{\theta(x)} x. \tag{14}$$

Let $\pi = [\chi]$ and let $\gamma \colon G \to \mathbb{C}$ be defined by

$$\gamma(g) = \sum_{x \in G} \chi(xgx^{-1})$$

where $\chi(g)$ is as in (6). Then clearly γ is a class function of G. By Theorem 2.1, $\mathcal{C}_\pi$ has a primitive idempotent ϵ satisfying (12). Then for $\theta \in \operatorname{Irr} G, \theta \neq \pi$,

$$\begin{aligned} 0 &= \epsilon\eta_\theta \quad \text{(by (13))} \\ &= \epsilon\eta_\theta\epsilon \\ &= \frac{1}{|G|}\sum_{g\in G}\overline{\theta(g)}\chi(g)\epsilon \quad \text{(by (6), (12), (14))}. \end{aligned}$$

Since θ is a class function we see that

$$\sum_{g\in G}\overline{\theta(g)}\chi(xgx^{-1}) = 0, \qquad x \in G.$$

Summing over all $x \in G$, we see that $\langle\theta, \gamma\rangle = 0$. It follows that γ is a scalar multiple of π. Since $\gamma(1) = |G|$, we see that $\pi = \frac{m}{|G|}\gamma = \xi$. This completes the proof. □

We now obtain a sufficient condition for $\chi = (\chi_+, \chi_-)$ to be a weight. Let

$$\epsilon_\chi = \sum_{g\in G}\chi(g)g^{-1}. \tag{15}$$

Theorem 3.2. *Suppose ϵ_χ has rank 1, i.e.* $\dim \epsilon_\chi\mathbb{C}G\epsilon_\chi = 1$. *Then χ is a weight and $\epsilon = \frac{\deg[\chi]}{|G|}\epsilon_\chi$ is a primitive idempotent of $\mathbb{C}G$. The corresponding representation of $\mathcal{A}_\chi$ is obtained by sending e to ϵ.*

Proof. Let

$$\epsilon_1 = \frac{1}{|P|}\sum_{p\in P}\chi_+(p)p^{-1}, \qquad \epsilon_2 = \frac{1}{|P|}\sum_{q\in P^-}\chi_-(q)q^{-1}. \tag{16}$$

Then

$$\epsilon_1^2 = \epsilon_1, \quad \epsilon_2^2 = \epsilon_2, \qquad \epsilon_1\epsilon_2 = \frac{|P\cap P^-|}{|P|\,|P^-|}\epsilon_\chi. \tag{17}$$

By (11),

$$\epsilon_2\epsilon_1 = \epsilon_2^*\epsilon_1^* = (\epsilon_1\epsilon_2)^*.$$

So by (14), $\epsilon_1\epsilon_2, \epsilon_2\epsilon_1$ are rank 1 elements of $\mathcal{C}_\pi$ for some $\pi \in \operatorname{Irr} G$. Since $\mathcal{C}_\pi$ is a simple algebra, the rank 1 elements of $\mathcal{C}_\pi$ form a $\mathcal{J}$-class of the multiplicative semigroup of $\mathcal{C}_\pi$. So by [4, Chapter 3], if a, b, c are elements of rank 1 in $\mathbb{C}_\pi$, then

$$ab, bc \neq 0 \Rightarrow abc \neq 0. \tag{18}$$

Let $p \in P, q_1, q_2 \in P^-$. Then since χ_+, χ_- agree on $P \cap P^-$,

$$q_1pq_2 = 1 \Rightarrow p = q_1^{-1}q_2^{-1} \in P\cap P^- \Rightarrow \chi_-(q_1)\chi_+(p)\chi_-(q_2) = 1.$$

It follows that the coefficient of 1 in $\epsilon_2\epsilon_1\epsilon_2$ is non-zero. Hence $\epsilon_2\epsilon_1\epsilon_2 \neq 0$. Similarly $\epsilon_1\epsilon_2\epsilon_1 \neq 0$. So by (18),

$$(\epsilon_1\epsilon_2)^2 = (\epsilon_1\epsilon_2)(\epsilon_2\epsilon_1)(\epsilon_1\epsilon_2) \neq 0.$$

By (17), $\epsilon_\chi^2 \neq 0$. Since ϵ_χ has rank 1, $\epsilon_\chi^2 = \alpha\epsilon_\chi$ for some $\alpha \in \mathbb{C}, \alpha \neq 0$. So $\epsilon = \frac{1}{\alpha}\epsilon_\chi$ is a primitive idempotent of $\mathcal{C}_\pi$ The linear operator $E\colon \mathbb{C}G \to \mathbb{C}G$ given by

$$E(x) = \epsilon x, \qquad x \in \mathbb{C}G,$$

is idempotent of rank $\deg \pi$. By (15), E has trace $\frac{|G|}{\alpha}$. Hence $\alpha = \frac{|G|}{\det \pi}$. By (16), (17),

$$p\epsilon = \chi_+(p)\epsilon, \quad eq = \chi_-(q)\epsilon \quad \text{for } p \in P,\ q \in P^-. \tag{19}$$

Let $x \in G, p_1, p_2 \in P, q_1, q_2 \in P^-$. Then

$$p_1 q_1 x p_2 q_2 = 1 \Rightarrow x^{-1} = p_2 q_2 p_1 q_1.$$

Hence the coefficient of 1 in $\epsilon x \epsilon$ is equal to the coefficient of x^{-1} in $\epsilon^2 = \epsilon$. Since $\epsilon x \epsilon$ is a scalar multiple of ϵ, we see by (15) that

$$\epsilon x \epsilon \neq 0 \Rightarrow x \in P^- P. \tag{20}$$

By (19), (20) and Theorem 2.1, χ is a weight and $[\chi] = \pi$. This completes the proof. □

Corollary 3.3. *If $\langle \chi_+ \uparrow G, \chi_- \uparrow G \rangle = 1$, then $\chi = (\chi_+, \chi_-)$ is a weight of the representation corresponding to the common component of $\chi_+ \uparrow G$ and $\chi_- \uparrow G$.*

Proof. Let π be the common component of $\chi_+ \uparrow G$ and $\chi_- \uparrow G$. Let $\mathcal{C}_\pi, \eta_\pi$ be as in (13), (14), respectively. Since $\langle \chi_+ \uparrow G, \pi \rangle = 1 = \langle \chi_- \uparrow G, \pi \rangle$, we see by Frobenius reciprocity that $\eta_\pi \epsilon_1$ and $\eta_\pi \epsilon_2$ are primitive idempotents and that $\eta_\theta \epsilon_1 \epsilon_2 = 0$ if $\theta \in \operatorname{Irr} G$, $\theta \neq \pi$. Hence

$$\epsilon_1 \epsilon_2 = \eta_\pi \epsilon_1 \epsilon_2 = (\eta_\pi \epsilon_1)(\eta_\pi \epsilon_2)$$

is a rank 1 element of $\mathcal{C}_\pi$. Hence by (17), ϵ_χ has rank 1. By Theorem 3.2, χ is a weight. □

Example 3.4. Let $G = S_n$, the symmetric group of degree n. Let α, α' be dual partitions of n, and $[\alpha]$ the associated irreducible representation of S_n. Let $\mathcal{S}_\alpha, S_{\alpha'}$ be the associated Young subgroups with $S_\alpha \cap S_{\alpha'} = 1$. Let χ_+ be the trivial character on S_α and let χ_- be the alternating character of S_α. By [8, Theorem 2.1.3], $\langle \chi_+ \uparrow G, \chi_- \uparrow G \rangle = 1$ with the character of $[\alpha]$ being the common component. Hence by Corollary 3.3, (χ_+, χ_-) is a weight for $[\alpha]$. We also note that ϵ_χ as in (15) is the associated Young Symmetrizer; cf. [6, Section 2], [7, Section 4.1].

Example 3.5. Let G be a finite Lie type group with opposite Borel subgroups B, B^-. Let U denote the unipotent radical of B. Let U' denote the product of the positive root subgroups not corresponding to the simple roots. Let χ_+ be a linear character of U lifted from a non-degenerate linear character of U/U', as in [2, Chapter 8]. So $\chi_+ \uparrow G$ is the Gel′fand-Graev character of G. Hence the irreducible components of $\chi_+ \uparrow G$ have multiplicity one and are the regular characters of G. Let χ_- be the trivial character of B^-. Then the Steinberg character occurs with multiplicity one in $\chi_- \uparrow G$ and the irreducible components of $\chi_- \uparrow G$ are among the unipotent characters of G. By [2, Chapter 12], the Steinberg character is the only character of G that is both regular and unipotent. Hence $\langle \chi_+ \uparrow G, \chi_- \uparrow G \rangle = 1$ with the Steinberg character being the common component. By Corollary 3.3, (χ_+, χ_-) is a weight for the Steinberg representation of G.

Example 3.6. Let $G = GL_n(\mathbb{F}_q)$. We can combine Examples 3.4 and 3.5 to obtain weights for all the unipotent representations of G. Let α, α' be dual partitions of n and let $S_\alpha, S_{\alpha'}$ be associated Young subgroups with $S_\alpha \cap S_{\alpha'} = 1$. Let $P_\alpha, P_{\alpha'}$ be parabolic subgroups with Weyl groups $S_\alpha, S_{\alpha'}$, respectively. We may assume that P_α consists of block upper triangular matrices. Let L_α denote the Levi subgroup of block diagonal matrices of P_α. Let U denote the group of unipotent upper triangular matrices. Let U' denote the normal subgroup of U generated by root subgroups with the root not corresponding to a simple reflection of S_α. Let χ_+ be a linear character of U obtained by lifting a non-degenerate linear character of the abelian group U/U' to U. Thus $\chi_+ \uparrow P_\alpha$ is equal to the Gel′fand-Graev character of L_α lifted to P_α.

Since $S_\alpha \cap S_{\alpha'} = 1$, $P_{\alpha'} \cap L_\alpha$ is a Borel subgroup of L_α. Hence, for some $\sigma \in S_\alpha$, $\sigma^{-1}(P_{\alpha'} \cap L_\alpha)\sigma$ consists of lower triangular matrices. Let $P_\alpha^- = \sigma^{-1} P_{\alpha'} \sigma$ and let χ_- be the trivial character on P_α^-. Then

$$\chi_+|_{U \cap P_\alpha^-} = \chi_-|_{U \cap P_\alpha^-}, \qquad \langle \chi_+ \uparrow G, \chi_- \uparrow G \rangle = 1.$$

By Corollary 3.3, $\chi_\alpha = (\chi_+, \chi_-)$ is a weight. Thus $[\chi_\alpha]$, α a partition of n, are all the unipotent representations of G.

Let G be a finite group of Lie type. The unipotent characters of G have been studied in detail by Lusztig [9]. Finding weights for these unipotent representations remains an open problem.

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