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A NONALGEBRAIC ATTRACTOR IN $\mathbf{P}^2$

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ABSTRACT. We construct a nonalgebraic attractor for a holomorphic mapping on $\mathbf{P}^2$. The construction uses ideas from one-dimensional complex dynamics.

0. INTRODUCTION

Attractors for dynamical systems are interesting, since they, in some sense, reflect the physically observable features of the dynamics. Holomorphic dynamical systems have a certain rigidity which imposes restrictions on the possible geometry of attractors. In the case of holomorphic mappings on complex projective space $\mathbf{P}^2$, Fornæss and one of the present authors showed, among other things, that the complement of an (infinite) attractor is pseudoconvex [FW]. It is interesting to analyze what kinds of attractors are possible. Easy examples of attractors are points and lines. In this paper we will construct an attractor which is not algebraic. As far as we know, it is the first example of its kind. The construction uses ideas from one-dimensional dynamics and the dynamics on the attractor is related to a critically finite rational map on the Riemann sphere.

We recall the definition of an attractor in the sense of Ruelle [R]. Let (X, d) be a compact metric space and let $f : X \to X$ be a surjective continuous map. A sequence $(x_i)_{0\leq i\leq k}$ of points in X is an *ϵ-pseudoorbit* if $d(fx_i, x_{i+1}) \leq \epsilon$ for all i. If $x, y \in X$, then we write $x \succ y$ if for every $\epsilon > 0$ there exists an ϵ-pseudoorbit $(x_i)_{0\leq i\leq k}$ with $x_0 = x$ and $x_k = y$. The preorder $\succ$ is reflexive and transitive. Say that $x \sim y$ if $x \succ y$ and $y \succ x$. This defines an equivalence relation on X, the equivalence classes of which are closed subsets of X. The ordering $\succ$ induces an ordering between equivalence classes. An *attractor* for f is a minimal equivalence class.

We will not work with pseudoorbits but with attracting sets. A compact set $K \subset X$ is an *attracting set* if K has a neighborhood U such that $fU \subset\subset U$ and $K = \bigcap_{n\geq 0} f^n U$. It is easy to see that if K is an attracting set and $f|_K$ is transitive (i.e. has a dense orbit), then K is an attractor.

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1. The attractor

We will consider the family of maps

$$f_\lambda[z:w:t] = [z^2 - 2w^2 : z^2 : t^2 + \lambda z^2]$$

for small $\lambda \in \mathbf{C}$. The purpose of this paper is to prove

Proposition A. *The map f_λ has a nonalgebraic attractor K_λ for all $\lambda \in \mathbf{C}$ with $0 < |\lambda| < 1/20$.*

Let us first give the idea of the proof. Note that the map f_0 has an attracting set $\Pi = \{t = 0\}$, the line at infinity. The induced map on $\Pi \simeq \mathbf{P}^1$ is given by $g[z:w] = [z^2 - 2w^2 : z^2]$, which is a critically finite rational map. Thus the Julia set of g is all of Π (see [CG]), so g is topologically mixing on Π and Π is an attractor for f_0.

It is not difficult to see that any small perturbation of f_0 will have an attracting set K near Π. In general, K will not be an attractor. However, the specific maps f_λ have the property that they preserve the set of lines through the point $[0:0:1]$. This set is naturally identified with $\mathbf{P}^1$ and the induced rational map on $\mathbf{P}^1$ is exactly g above. Using this, we will see that f_λ is topologically mixing on K_λ, so that K_λ is an attractor.

We will show that K_λ is nonalgebraic for $\lambda \neq 0$ by some explicit computations and an analysis of the dynamics near the saddle fixed point $[1:-1:t_\lambda]$.

Let us now begin the actual proof of Proposition A. We will use the following two computational results, the proofs of which are left to the reader.

Lemma 1. *If $0 < s \leq 1/20$, then the polynomial $x^2 + (2s - 1/4)x + s^2$ has two real roots α and β which satisfy $0 < \alpha < 9s^2 < 1/9 < \beta$.*

Lemma 2. *Suppose $\lambda \in \mathbf{C}$ satisfies $0 < |\lambda| < 1/20$. Let $t_\lambda = (-1 + \sqrt{1 - 4\lambda})/2$, where we choose the branch of the square root that maps 1 to 1. Then $|t_\lambda| < \sqrt{18}|\lambda|$ and $|\lambda - t_\lambda| > 18|\lambda|^2$.*

The following lemma describes the rough mapping properties of f_λ near Π.

Lemma 3. *For $\rho > 0$, let $U_\rho \subset \mathbf{P}^2$ be the neighborhood of Π defined by*

$$U_\rho = \{[z:w:t] : |t| < \rho|(z,w)|\}.$$

Assume $0 < |\lambda| < 1/20$ and $3|\lambda| \leq \rho \leq 1/3$. Then $f_\lambda U_\rho \subset\subset U_\rho$ and $f_\lambda^n U_\rho \subset\subset U_{3|\lambda|}$ for large n.

Proof. Write $f_\lambda^n[z:w:t] = [z_n : w_n : t_n]$. One readily checks that

$$|(z^2 - 2w^2, z^2)| \geq \frac{1}{2}|(z,w)|^2$$

for all (z,w). This implies

$$\begin{aligned}\frac{|t_n|^2}{|z_n|^2 + |w_n|^2} &\leq 4\frac{|t_{n-1}^2 + \lambda z_{n-1}^2|^2}{(|z_{n-1}|^2 + |w_{n-1}|^2)^2} \\ &\leq 4\frac{|t_{n-1}|^4}{(|z_{n-1}|^2 + |w_{n-1}|^2)^2} + 8|\lambda|\frac{|t_{n-1}|^2}{|z_{n-1}|^2 + |w_{n-1}|^2} + 4|\lambda|^2.\end{aligned}$$

Write $x_n = |t_n|^2/(|z_n|^2 + |w_n|^2)$. We then have

$$x_n \leq 4\left(x_{n-1}^2 + 2|\lambda|x_{n-1} + |\lambda|^2\right).$$

Since $0 < |\lambda| \le 1/20$, it follows from Lemma 1 with $s = |\lambda|$ that if $9|\lambda|^2 \le x_0 \le 1/9$, then $x_n < x_{n-1}$ for all $n \ge 1$. Iterating this implies that $x_n < 9|\lambda|^2$ for large n. This proves the lemma. □

In the sequel we will write $U = U_{1/3}$.

Lemma 4. *Let* $|\lambda| < 1/20$ *and define*

$$K_\lambda = \bigcap_{n\ge 0} f_\lambda^n U.$$

Then $K_\lambda \subset U_{3|\lambda|}$ *and* $f_\lambda K_\lambda = K_\lambda$. *Further,* f *is topologically mixing on* K_λ, *so* K_λ *is an attractor for* f_λ.

Proof. Everything except the last statement follows from Lemma 3. Before proving that f_λ is topologically mixing on K_λ we note some dynamical properties pertinent to the special form of f_λ.

Let $\Pi = \{t = 0\} \simeq \mathbf{P}^1$ be the hyperplane at infinity and let g be the rational map on Π given by $g[z : w] = [z^2 - 2w^2 : z^2]$. The projection $\pi[z : w : t] = [z : w]$ semiconjugates f_λ to g: $g \circ \pi = \pi \circ f_\lambda$.

For $a \in \Pi$, let $L_a = \overline{\pi^{-1}(a)}$ be the line in $\mathbf{P}^2$ passing through a and $[0 : 0 : 1]$, and let $V_a := U \cap L_a$. Then V_a is a disk in L_a centered at a, and Lemma 3 shows that $f_\lambda V_a \subset\subset V_{ga}$ for all $a \in \Pi$. Using the hyperbolic metric on V_a we see that the diameter (in the Fubini-Study metric on $\mathbf{P}^2$) of the set $f_\lambda^n V_a$ tends to zero, uniformly in a, as $n \to \infty$.

Let $\hat\Pi$ be the set of histories in Π under g, i.e. the set of sequences $\hat a = (a_i)_{i\le 0}$ such that $a_i \in \Pi$ and $ga_i = a_{i+1}$ for all i. Then $\hat\Pi$ is a closed subset of $\Pi^{\mathbf{Z}}$ and is therefore compact. By the remark above, if $\hat a \in \hat\Pi$, then the intersection $\bigcap_{i\le 0} f_\lambda^{-i} V_{a_i}$ is a single point, which we denote by $h_\lambda(\hat a)$. We claim that

$$K_\lambda = \{h_\lambda(\hat a) : \hat a \in \hat\Pi\}.$$

Indeed, $h_\lambda(\hat a) \in f_\lambda^n U$ for all $n \ge 0$ so $h_\lambda(\hat a) \in K_\lambda$. Conversely, if $x \in K_\lambda$, then x has a history $\hat x = (x_i)_{i\le 0}$ in K_λ. Let $a_i = \pi(x_i)$. Then $\hat a = (a_i) \in \hat\Pi$ and $x = h_\lambda(\hat a)$.

We now prove that f_λ is topologically mixing on K_λ. Let Ω_1 and Ω_2 be two open sets with $\Omega_i \subset U$ and $\Omega_i \cap K_\lambda \ne \emptyset$ for $i = 1, 2$. We will show that $f_\lambda^n \Omega_1 \cap \Omega_2 \ne \emptyset$ for all sufficiently large n. Pick $x \in \Omega_2 \cap K_\lambda$ and find $\hat a \in \hat\Pi$ such that $x = h_\lambda(\hat a)$. We may find $k \ge 0$ such that $f_\lambda^k V_{a_{-k}} \subset\subset \Omega_2$. By continuity there is an open neighborhood ω of a_{-k} in Π such that if $a \in \omega$, then $f_\lambda^k V_a \subset \Omega_2$. Let $\omega_1 = \pi\Omega_1 \subset \Pi$. Since g is topologically mixing on Π, there exists $N \ge 0$ such that if $n \ge N$, then $g^n\omega_1 \cap \omega \ne \emptyset$. It follows that $f_\lambda^n \Omega_1 \cap \Omega_2 \ne \emptyset$ for $n \ge N + k$. Thus f_λ is topologically mixing on K_λ. □

It remains to be seen that K_λ is nonalgebraic when $0 < |\lambda| < 1/20$. This follows immediately from the following proposition.

Proposition 5. *If* $0 < |\lambda| < 1/20$, *then there is no algebraic curve* $W \subset U$ *which satisfies* $f_\lambda W \subset W$.

Proof. Suppose there is such a curve W. We may assume that $f_\lambda W = W$, because otherwise we may replace W by the algebraic curve $W' = \bigcap_{n\ge 0} f_\lambda^n W$. By Lemma 3 we must have $W \subset U_{3|\lambda|}$.

Let $L = L_{[1:-1]}$ be the invariant line $(z = -w)$, and let $L' = L_{[1:1]} = (z = w)$, the strict preimage of L. The restrictions $f_\lambda|_L$ and $f_\lambda|_{L'}$ are given by

$$[1 : -1 : t] \to [1 : -1 : g_\lambda(t)] \quad \text{and} \quad [1 : 1 : t] \to [1 : -1 : g_\lambda(t)],$$

respectively, where $g_\lambda(t) = -(t^2 + \lambda)$. Also,

$$U_{3|\lambda|} \cap L = \{[1 : -1 : t] : |t| < \sqrt{18}|\lambda|\} \text{ and } U_{3|\lambda|} \cap L' = \{[1 : 1 : t] : |t| < \sqrt{18}|\lambda|\}.$$

Let p_λ be the fixed point $[1 : -1 : t_\lambda] \in L$, where $t_\lambda = (-1 + \sqrt{1 - 4\lambda})/2$, and where we use the branch of the square root which maps 1 to 1. Then $p_\lambda \in U_{3|\lambda|}$ by Lemma 2. By Lemma 3 the disk $U_{3|\lambda|} \cap L$ is mapped strictly into itself by f_λ and is therefore contained in the basin of attraction (for $f_\lambda|_L$) to the fixed point p_λ. Since $f_\lambda W \subset W$ and $\emptyset \neq W \cap L \subset U_{3|\lambda|} \cap L$, therefore, we must have $p_\lambda \in W$ and every point in the (finite) set $W \cap L$ is preperiodic to p_λ. We have the following cases.

(i) $W \cap L = \{p_\lambda\}$.

(ii) $W \cap L$ contains more than one point.

The fixed point p_λ is a hyperbolic saddle point for f_λ. Since $f_\lambda W \subset W$, this implies that any local irreducible branch of W at p_λ must coincide with either the unstable or the stable manifold of f_λ at p_λ. But the stable manifold is exactly L, and no branch of W can be contained in L. Thus the germ of W at p_λ coincides with the local unstable manifold at p_λ. In particular it is transverse to L at p_λ.

In case (i) this implies that W is a line. However, a direct computation shows that U contains no invariant lines unless $\lambda = 0$. Thus we are left with case (ii). Using the fact that each point in $W \cap L$ is preperiodic to p_λ, we see that $W \cap L$ contains the preimage $p'_\lambda := [1 : -1 : -t_\lambda]$ of p_λ. Since $f_\lambda W = W$, the point p'_λ has a preimage in W, and this must be of the form $[1 : \pm 1 : \pm u_\lambda]$, where u_λ satisfies $-u_\lambda^2 - \lambda = -t_\lambda$. Thus $|u_\lambda| = \sqrt{|\lambda - t_\lambda|} > \sqrt{18}|\lambda|$ by Lemma 2 and this implies that $[1 : \pm 1 : \pm u_\lambda] \notin U_{3|\lambda|}$. This contradicts $W \subset U_{3|\lambda|}$ and completes the proof of Proposition 5. ☐

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