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# OSCILLATION CRITERIA FOR DELAY EQUATIONS

M. KON, Y. G. SFICAS, AND I. P. STAVROULAKIS

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*Dedicated to Professor V. A. Staikos on the occasion of his 60th birthday*

Abstract. This paper is concerned with the oscillatory behavior of first-order delay differential equations of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+), \mathbb{R}^+ = [0, \infty), \tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. Let the numbers $k$ and $L$ be defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

It is proved here that when $L < 1$ and $0 < k \leq \frac{1}{e}$ all solutions of Eq. (1) oscillate in several cases in which the condition

$$L > 2k + \frac{2}{\lambda_1} - 1$$

holds, where $\lambda_1$ is the smaller root of the equation $\lambda = e^{k\lambda}$.

## 1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions of the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)), \tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [1]–[26] and the references cited therein.

By a solution of Eq. (1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory.

The first systematic study for the oscillation of all solutions of Eq. (1) was made by Myshkis. In 1950 [23] he proved that every solution of Eq. (1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty, \quad \liminf_{t\to\infty}[t - \tau(t)] \cdot \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{$C_1$}$$

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In 1972, Ladas, Lakshmikantham and Papadakis [18] proved that the same conclusion holds if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 1. \tag{$C_2$}$$

In 1979 Ladas [17] and in 1982 Koplatadze and Chanturija [13] improved $(C_1)$ to

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{$C_3$}$$

Concerning the constant $\frac{1}{e}$ in $(C_3)$, it is to be pointed out that if the inequality

$$\int_{\tau(t)}^{t} p(s)ds \le \frac{1}{e}$$

holds eventually, then, according to a result in [13], (1) has a non-oscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [19] and in 1984 Fukagai and Kusano [10] established oscillation criteria (of the type of conditions $(C_2)$ and $(C_3)$) for Eq. (1) with *oscillating* coefficient $p(t)$.

It is obvious that there is a gap between the conditions $(C_2)$ and $(C_3)$ when the limit

$$\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [9] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq. (1). Their result, when formulated in terms of the numbers $k$ and $L$ defined by

$$k = \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad L = \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds,$$

says that all the solutions of Eq. (1) are oscillatory, if $0 < k \le \frac{1}{e}$ and

$$L > 1 - \frac{k^2}{4}. \tag{$C_4$}$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$. In 1991 Jian Chao [2] derived the condition

$$L > 1 - \frac{k^2}{2(1-k)}, \tag{$C_5$}$$

while in 1992 Yu and Wang [25] and Yu, Wang, Zhang and Qian [26] obtained the condition

$$L > 1 - \frac{1-k-\sqrt{1-2k-k^2}}{2}. \tag{$C_6$}$$

In 1990 Elbert and Stavroulakis [7] and in 1991 Kwong [16], using different techniques, improved $(C_4)$, in the case where $0 < k \le \frac{1}{e}$, to the conditions

$$L > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \tag{$C_7$}$$

and

$$L > \frac{\ell n\lambda_1 + 1}{\lambda_1}, \tag{$C_8$}$$

respectively, where $\lambda_1$ is the smaller root of the equation

$$\lambda = e^{k\lambda}. \tag{2}$$

In 1994 Koplatadze and Kvinikadze [14] improved ($C_6$), while in 1996 Philos and Sficas [24] and Jaroš and Stavroulakis [11] derived the conditions

$$L > 1 - \frac{k^2}{2(1-k)} - \frac{k^2}{2}\lambda_1 \tag{$C_9$}$$

and

$$L > \frac{\ell n\lambda_1 + 1}{\lambda_1} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \tag{$C_{10}$}$$

respectively, where $\lambda_1$ is the smaller root of Eq. (2).

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1993 by Elbert and Stavroulakis [8] and in 1995 by Kozakiewicz [15], Li [21], [22] and by Domshlak and Stavroulakis [5].

The purpose of this paper is to improve the methods previously used to show that in several cases the conditions ($C_2$) and ($C_4$)–($C_{10}$) may be weakened to

$$L > 2k + \frac{2}{\lambda_1} - 1, \tag{$C_{11}$}$$

where $\lambda_1$ is the smaller root of the equation $\lambda = e^{k\lambda}$.

It is to be noted that as $k \to 0$, then all conditions ($C_4$)–($C_{10}$) and also our condition ($C_{11}$) reduce to the condition ($C_2$). However the improvement is clear as $k \to \frac{1}{e}$. For illustrative purposes, we give the values of the lower bound on $L$ under these conditions when $k = \frac{1}{e}$:

| | |
|---|---|
| ($C_2$): | 1.000000000 |
| ($C_4$): | 0.966166179 |
| ($C_5$): | 0.892951367 |
| ($C_6$): | 0.863457014 |
| ($C_7$): | 0.845181878 |
| ($C_8$): | 0.735758882 |
| ($C_9$): | 0.709011646 |
| ($C_{10}$): | 0.599215896 |
| ($C_{11}$): | 0.471517764 |

We see that our condition ($C_{11}$) essentially improves all the known results in the literature.

## 2. Main results

In what follows we will denote by $k$ and $L$ the lower and upper limits of the average $\int_{\tau(t)}^{t} p(s)\,ds$ as $t\to\infty$, respectively, i.e.

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$$

and

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

Set

$$w(t) = \frac{x(\tau(t))}{x(t)}.$$

We begin with the preliminary analysis of asymptotic behavior of the function $w(t)$ for a possible nonoscillatory solution $x(t)$ of Eq. (1) in the case that $k \le \frac{1}{e}$. For this purpose, assume that (1) has a solution $x(t)$ which is positive for all large $t$. Dividing first Eq. (1) by $x(t)$ and then integrating it from $\tau(t)$ to $t$ leads to the integral equality

$$w(t) = \exp \int_{\tau(t)}^{t} p(s)w(s)ds \tag{3}$$

which holds for all sufficiently large $t$.

For the next lemma see [11].

**Lemma 1.** *Suppose that $k > 0$ and Eq. (1) has an eventually positive solution $x(t)$. Then $k \le 1/e$ and*

$$\lambda_1 \le \liminf_{t\to\infty} w(t) \le \lambda_2,$$

*where $\lambda_1$ is the smaller and $\lambda_2$ the greater root of the equation $\lambda = e^{k\lambda}$.*

*Proof.* Let $\alpha = \liminf_{t\to\infty} w(t)$. From (3) we have for sufficiently large $t$ that

$$\alpha \ge \exp k\alpha,$$

which is impossible if $k > 1/e$, since a simple calculus argument shows that in this case $\lambda < e^{k\lambda}$ for all $\lambda$. This implies that (1) has no eventually positive solution if $k > 1/e$. On the other hand, if $0 < k \le 1/e$, then $\lambda = e^{k\lambda}$ has roots $\lambda_1 \le \lambda_2$ (with equality $\lambda_1 = \lambda_2 = e$ if and only if $k = 1/e$), and $\alpha \ge e^{k\alpha}$ if and only if $\lambda_1 \le \alpha \le \lambda_2$.

**Lemma 2.** *Let $0 < k \le \frac{1}{e}$ and $x(t)$ be an eventually positive solution of Eq. (1). Assume that there exists $\theta > 0$ such that*

$$\int_{\tau(u)}^{\tau(t)} p(s)ds \;\ge\; \theta \int_{u}^{t} p(s)ds \quad for \;\; all \;\; \tau(t) \le u \le t. \tag{4}$$

*Then*

$$\limsup_{t\to\infty} w(t) \le \frac{2}{1-k-\sqrt{(1-k)^2-4A}}, \tag{5}$$

*where $A$ is given by*

$$A = \frac{e^{\lambda_1\theta k} - \lambda_1\theta k - 1}{(\lambda_1\theta)^2}. \tag{6}$$

*Proof.* Let $\delta : 0 < \delta < k$ be any number arbitrarily close to $k$ and $T > t_0$ large enough so that $\tau(t) > t_0$ and $\int_{\tau(t)}^{t} p(s)ds > \delta$ for every $t \ge T$. Let $t \ge T$ and $T_1 \equiv T_1(t) > t : \tau(T_1) = t$. Since $\int_{t}^{T_1} p(s)ds > \delta$ there exists $T_1 > t_1 \equiv t_1(t) > t$

such that

$$\int_t^{t_1} p(s)ds = \delta.$$

Integrating (1) from $t$ to $t_1$, we obtain

$$x(t) = x(t_1) + \int_t^{t_1} p(s)\, x\,(\tau(s))ds,$$

while integrating from $\tau(s)$ to $t$ for $s < t_1$, we have

$$x(\tau(s)) = x(t) + \int_{\tau(s)}^{t} p(u)\, x\,(\tau(u))\, du.$$

Combining the last two equalities, we obtain

$$x(t) = x(t_1) + \int_t^{t_1} p(s)\left[x(t) + \int_{\tau(s)}^{t} p(u)\, x\,(\tau(u))\, du\right] ds. \tag{7}$$

Let $0 < \lambda < \lambda_1$. Then the function

$$\varphi(t) = x(t)\, e^{\lambda \int_{t_0}^{t} p(s)ds}, \quad t \geq a,$$

is decreasing for appropriate $a \geq t_0$ since $x(t)$ is also decreasing. Indeed, by Lemma 1,

$$\frac{x(\tau(t))}{x(t)} > \lambda$$

for all sufficiently large $t$, and consequently

$$0 = x'(t) + p(t)x(\tau(t)) \geq x'(t) + \lambda\, p(t)x(t)$$

which implies $\varphi'(t) \leq 0$ for sufficiently large $t$.

Substituting into (7), we derive for sufficiently large $t$

$$x(t) \geq x(t_1) + \delta x(t) + \varphi(\tau(t)) \int_t^{t_1} p(s)\left(\int_{\tau(s)}^{t} p(u)\, e^{-\lambda \int_{t_0}^{\tau(u)} p(\xi)\, d\xi} du\right) ds$$

$$= x(t_1) + \delta x(t) + \varphi(\tau(t))\, e^{-\lambda \int_{t_0}^{\tau(t)} p(s)\, ds} \int_t^{t_1} p(s)\left(\int_{\tau(s)}^{t} p(u)\, e^{\lambda \int_{\tau(u)}^{\tau(t)} p(\xi) d\xi} du\right) ds$$

and therefore

$$x(t) \geq x(t_1) + \delta x(t) + x(\tau(t)) \int_t^{t_1} p(s)\left(\int_{\tau(s)}^{t} p(u)\, e^{\lambda \int_{\tau(u)}^{\tau(t)} p(\xi) d\xi} du\right) ds. \tag{8}$$

In view of (4), we obtain

$$\int_{\tau(s)}^{t} p(u)\, e^{\lambda \int_{\tau(u)}^{\tau(t)} p(\xi) d\xi} du \geq \int_{\tau(s)}^{t} p(u)\, e^{\lambda\theta \int_u^t p(\xi) d\xi} du$$

$$= \frac{1}{\lambda\theta}\left[e^{\lambda\theta \int_{\tau(s)}^{t} p(\xi) d\xi} - 1\right].$$

Thus

$$\begin{aligned}\int_t^{t_1} p(s)\left(\int_{\tau(s)}^{t} p(u)\, e^{\lambda \int_{\tau(u)}^{\tau(t)} p(\xi)d\xi} du\right) ds \;&\geq\; -\frac{\delta}{\lambda\theta} + \frac{1}{\lambda\theta}\int_t^{t_1} p(s) e^{\lambda\theta \int_{\tau(s)}^{t} p(\xi)d\xi} ds \\ &= -\frac{\delta}{\lambda\theta} + \frac{1}{\lambda\theta}\int_t^{t_1} p(s) e^{\lambda\theta \int_{\tau(s)}^{s} p(\xi)d\xi - \lambda\theta \int_t^s p(\xi)d\xi} ds \\ &\geq -\frac{\delta}{\lambda\theta} + \frac{1}{\lambda\theta}\, e^{\lambda\theta\delta}\int_t^{t_1} p(s) e^{-\lambda\theta \int_t^s p(\xi)d\xi} ds \\ &= -\frac{\delta}{\lambda\theta} + \frac{e^{\lambda\theta\delta}}{(\lambda\theta)^2}\left[1 - e^{-\lambda\theta \int_t^{t_1} p(\xi)d\xi}\right] \\ &= -\frac{\delta}{\lambda\theta} + \frac{e^{\lambda\theta\delta}}{(\lambda\theta)^2}\left[1 - e^{-\lambda\theta\delta}\right] = -\frac{\delta}{\lambda\theta} + \frac{1}{(\lambda\theta)^2}\left(e^{\lambda\theta\delta} - 1\right),\end{aligned}$$

and (8) yields

$$x(t) \;\geq\; x(t_1) + \delta x(t) + A^* x(\tau(t)), \tag{9}$$

where

$$A^* = \frac{e^{\lambda\theta\delta} - \lambda\theta\delta - 1}{(\lambda\theta)^2}.$$

From (9), we have

$$x(t) \;\geq\; d_1\, x(\tau(t)),$$

where we have set

$$d_1 = \frac{A^*}{1-\delta}.$$

Observe that

$$x(t_1) \;\geq\; d_1\, x(\tau(t_1)) \;\geq\; d_1 x(t)$$

since $x(t)$ is decreasing, and therefore (9) yields

$$x(t) \;\geq\; d_2\, x(\tau(t)),$$

where

$$d_2 = \frac{A^*}{1 - d_1 - \delta}.$$

Following this iterative procedure (cf. [25], [26]), we obtain

$$x(t) \;\geq\; d_{n+1}\, x(\tau(t)),$$

where

$$d_{n+1} = \frac{A^*}{1 - d_n - \delta}, \quad n = 1, 2, \ldots.$$

It is easy to see that the sequence $\{d_n\}$ is strictly increasing and bounded.

Therefore $\lim_{n\to\infty} d_n = d$ exists and satisfies

$$d^2 - (1-\delta)d + A^* = 0.$$

Since $\{d_n\}$ is strictly increasing, it follows that

$$d = \frac{1 - \delta - \sqrt{(1-\delta)^2 - 4A^*}}{2}.$$

Thus, for all large $t$

$$\frac{x(t)}{x(\tau(t))} \;\geq\; \frac{1 - \delta - \sqrt{(1-\delta)^2 - 4A^*}}{2}$$

and since $0 < \delta < k$ is arbitrarily close to $k$, by letting $\lambda \to \lambda_1$ the last inequality leads to (5). The proof is complete.

*Remark* 1. Assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that

$$p(\tau(t))\ \tau'(t)\ \geq\ \theta\ \ p(t) \tag{10}$$

eventually for all $t$. Then it is easy to see that (10) implies (4). Indeed, the function

$$v(u) = \int_{\tau(u)}^{\tau(t)} p(s)ds - \theta \int_u^t p(s)ds, \quad \tau(t) \leq u \leq t,$$

satisfies the conditions

$$v(t)\ =\ 0,$$

and

$$v'(u) = \ -p(\tau(u))\tau'(u) + \theta p(u)\ \leq\ 0.$$

If $p(t) > 0$ eventually for all $t$ and

$$\liminf_{t\to\infty} \frac{p(\tau(t))\ \tau'(t)}{p(t)}\ =\ \theta_0 > 0,$$

then $\theta$ can be any number satisfying $0 < \theta < \theta_0$ . Besides the case $p(t)\ \equiv\ p > 0,\ \tau(t) = t - \tau$ or the case $\tau(t) = t - \tau$ and $p(t)$ is $\tau$-periodic, there exists a class of functions which satisfy (10).

**Theorem.** *Consider the differential equation (1) and let* $L < 1,\ 0 < k \leq \frac{1}{e}$ *and there exist* $\theta > 0$ *such that*

$$\int_{\tau(u)}^{\tau(t)} p(s)ds \geq \theta \int_u^t p(s)ds \quad for\ \ all \quad \tau(t) \leq u \leq t. \tag{4}$$

*Assume that*

$$L\ >\ \frac{\ell n\lambda_1 + 1}{\lambda_1}\ -\ \frac{1 - k - \sqrt{(1-k)^2 - 4A}}{2}, \tag{11}$$

*where* $\lambda_1$ *is the smaller root of the equation* $\lambda = e^{k\lambda}$ *and* $A$ *is given by (6). Then all solutions of Eq. (1) oscillate.*

*Proof.* Assume, for the sake of contradiction, that x(t) is an eventually positive solution of Eq. (1). Then, as in [11, Theorem 1], we obtain

$$L\ \leq\ \frac{\ell n\lambda_1 + 1}{\lambda_1}\ -\ M,$$

where

$$M\ = \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))}.$$

The last inequality, in view of Lemma 2, contradicts (11). The proof is complete. □

*Remark* 2. Observe that when $\theta = 1$, then

$$A\ =\ \frac{\lambda_1 - \lambda_1 k - 1}{\lambda_1^2}$$

and (11) reduces to

$$L\ >\ 2k + \frac{2}{\lambda_1}\ - 1. \tag{12}$$

In the case that $k = \frac{1}{e}$, then $\lambda_1 = e$ and (12) leads to

$$L \; > \; \frac{4}{e} \; - 1 \approx 0.471517764.$$

**Example.** Consider the delay differential equation

$$x'(t) + p\; x\left(t - a\sin^2\sqrt{t} - \frac{1}{pe}\right) \; = \; 0, \tag{13}$$

where $p > 0,\; a > 0$ and $pa \; = \; \frac{1}{2} - \frac{1}{e}$. Then

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} \; p\left(a\sin^2\sqrt{t} + \frac{1}{pe}\right) \; = \; \frac{1}{e}$$

and

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} \; p\left(a\sin^2\sqrt{t} + \frac{1}{pe}\right) \; = \; pa \; + \; \frac{1}{e} \; = \; \frac{1}{2}.$$

Thus, according to Remark 2, all solutions of Eq. (13) oscillate. Observe that none of the results mentioned in the introduction can be applied to this equation.

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Department of Mathematics, Boston University, Boston, Massachusetts 02215
*E-mail address*: mkon@math.bu.edu

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
*E-mail address*: ipstav@cc.uoi.gr