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A COUNTEREXAMPLE FOR H^∞ APPROXIMABLE FUNCTIONS

DANIEL SUÁREZ

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Abstract. Let $\mathbb{D}$ be the unit disk. We show that for some relatively closed set $F \subset \mathbb{D}$ there is a function f that can be uniformly approximated on F by functions of H^∞, but such that f cannot be written as $f = h + g$, with $h \in H^\infty$ and g uniformly continuous on F. This answers a question of Stray.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the open unit disk and $F \subset \mathbb{D}$ be a relatively closed set. For $0 < p \leq \infty$ let $A^p(F)$ be the space of functions that can be uniformly approximated on F by functions of the Hardy space H^p. Also, let $C_{ua}(F)$ denote the space of uniformly continuous functions on F that are analytic on its interior. Put

$$\hat{F} = \{z \in \mathbb{D} : |f(z)| \leq \sup_F |f| \text{ for all } f \in H^\infty\}.$$

A. Stray proved in [4] that if $0 < p < \infty$, then

$$A^p(F) = H^p|_F + C_{ua}(\hat{F})|_F. \tag{1}$$

He also showed that $A^\infty(F) \supset C_{ua}(\hat{F})$. Notice that this immediately implies the inclusion of the right member of (1) in $A^p(F)$ for every $0 < p \leq \infty$. The problem of whether the other inclusion holds for $p = \infty$ is posed in [4]. The purpose of this paper is to construct an example where the inclusion fails. That is, we will see that there exists a relatively closed set $F \subset \mathbb{D}$, with $F = \hat{F}$, such that some $f \in A^\infty(F)$ cannot be decomposed as $f = h+g$, with $h \in H^\infty$ and $g \in C_{ua}(F)$. More generally, for the example that we construct the above decomposition is not even possible for $g \in C_{ua}(F)$ and h in the Bloch space. In this section we fix notation and state some of the background that will be used in the construction of the example, given in Section 2. The last section is a short discussion showing that the counterexample works in a more general situation.

The maximal ideal space $\mathcal{M}$ of H^∞ is defined as the space of nontrivial multiplicative linear functionals on H^∞, provided with the weak $*$ topology. It is a compact Hausdorff space, and the Gelfand transform, $\hat{f}(x) = x(f)$ for $f \in H^\infty$ and $x \in \mathcal{M}$, establishes an isometric morphism from H^∞ into the uniform algebra of continuous functions on $\mathcal{M}$. Evaluations at points of the disk are in $\mathcal{M}$, so $\mathbb{D}$ is

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naturally imbedded as an open subset of $\mathcal{M}$. In addition, the corona theorem [1] states that $\mathbb{D}$ is dense in $\mathcal{M}$.

Suppose that $\varphi \in \mathcal{M} \setminus \mathbb{D}$. In [3] Hoffman defined a continuous map L_φ from $\mathbb{D}$ into $\mathcal{M} \setminus \mathbb{D}$ that generalizes the concept of Mobius transformation, $L_\omega(z) = (\omega - z)/(1 - \overline{\omega} z)$ for $\omega \in \mathbb{D}$. He showed that if (z_α) is a net in $\mathbb{D}$ that tends to φ, then $f \circ L_{z_\alpha}$ tends pointwise to $\hat{f} \circ L_\varphi \in H^\infty$ for every $f \in H^\infty$. The use of nets is imposed by the fact that $\mathcal{M}$ is not metrizable.

Let b be a Blaschke product with zero sequence $\{z_n\}$. If

$$\delta_n = \prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right|$$

satisfies $\delta(b) \stackrel{\text{def}}{=} \inf \delta_n > 0$, then b is called an interpolating Blaschke product and $\{z_n\}$ an interpolating sequence. If *in addition* $\lim \delta_n = 1$ when $n \to \infty$, then b is called a thin product and $\{z_n\}$ a thin sequence. The pseudohyperbolic metric on $\mathbb{D}$ is $\rho(z, \omega) = |(\omega - z)/(1 - \overline{\omega} z)|$. For $z \in \mathbb{D}$ and $0 < r < 1$ we write

$$K(z, r) = \{\omega \in \mathbb{D} : \rho(z, \omega) < r\} \quad \text{and} \quad \Delta(z, r) = \{\omega \in \mathbb{D} : |z - \omega| < r\}$$

for the open balls of center z and radius r with respect to the pseudohyperbolic and the euclidean metric, respectively. Simple geometrical considerations [2, p. 3] show that $K(z, r) = \Delta(c, R)$, where

$$c = \frac{1 - r^2}{1 - r^2 |z|^2} z \quad \text{and} \quad R = \frac{1 - |z|^2}{1 - r^2 |z|^2} r. \tag{2}$$

The following well-known result of Hoffman can be found in [3] or [2, pp. 404-405].

Lemma 1. *Let b be a Blaschke product with zero sequence $\{z_n\}$. If $\delta = \delta(b) > 0$, then there are $0 < \varepsilon(\delta), r(\delta) < 1$ such that*

(i) *$a(z) = (b(z) - \omega)/(1 - \overline{\omega} b(z))$ is an interpolating Blaschke product when $|\omega| < \varepsilon(\delta)$,*
(ii) *$\{z \in \mathbb{D} : |b(z)| < \varepsilon(\delta)\} = \bigcup_n V_n$, where $V_n \subset K(z_n, r(\delta))$, and*
(iii) *$b(V_n) = \Delta(0, \varepsilon(\delta))$ for every n.*

2. The example

Let $X \subset \mathbb{D}$ be any subset. We can think of X as contained in $\mathcal{M}$ or in the complex plane $\mathbb{C}$. For $G = \mathcal{M}$ or $\mathbb{C}$ we write $\operatorname{clos}_G X$ for the closure of X in the space G.

Let b be a thin product with zero sequence $0 \leq z_n < 1$ (for instance $z_n = 1 - n^{-n}$) and put $\delta = \delta(b)$. We fix some ε with $0 < \varepsilon < \varepsilon(\delta)$, where $\varepsilon(\delta)$ is given by Lemma 1. Consider the sets

$$\Omega_\varepsilon = \{\omega \in \mathbb{R} : \varepsilon/2 \leq |\omega| \leq \varepsilon\}$$

and

$$E = \{x \in \mathcal{M} : \hat{b}(x) \in \Omega_\varepsilon\}.$$

Then $F \stackrel{\text{def}}{=} E \cap \mathbb{D}$ is a relatively closed subset of $\mathbb{D}$ such that $\hat{F} = F$. In fact, suppose that $z \in \mathbb{D}$ is not in F. Then $b(z) \notin \Omega_\varepsilon$ and since Ω_ε is polynomially convex, there is a polynomial p such that $|p(b(z))| > \sup_{\Omega_\varepsilon} |p| = \sup_F |p \circ b|$. Since $p \circ b \in H^\infty$, then $z \notin \hat{F}$.

Furthermore, we claim that $E = \mathrm{clos}_{\mathcal{M}}F$. The continuity of $\hat{b}$ on $\mathcal{M}$ obviously implies that $E \supset \mathrm{clos}_{\mathcal{M}}F$. For the other inclusion take $x \in E$. Therefore $\hat{b}(x) \in \Omega_\varepsilon \subset \{z \in \mathbb{D} : |z| < \varepsilon(\delta)\}$, and (i) of Lemma 1 tells us that

$$a(z) = (b(z) - \hat{b}(x))/(1 - \overline{\hat{b}(x)}b(z))$$

is an interpolating Blaschke product. If $\{\omega_n\}$ denotes the zero sequence of a and $Z(\hat{a}) = \{y \in \mathcal{M} : \hat{a}(y) = 0\}$, then the fact that a is interpolating easily yields $\mathrm{clos}_{\mathcal{M}}\{\omega_n\} = Z(\hat{a})$ (see [2, VII, Ex. 4]). Besides, we have $b(\omega_n) = \hat{b}(x) \in \Omega_\varepsilon$ for all n, which means that $\{\omega_n\} \subset F$. Therefore $x \in Z(\hat{a}) = \mathrm{clos}_{\mathcal{M}}\{\omega_n\} \subset \mathrm{clos}_{\mathcal{M}}F$, and the inclusion is proved.

Let $\varphi \in \mathrm{clos}_{\mathcal{M}}\{z_n\} \setminus \mathbb{D}$. Since $\{z_n\}$ is a thin sequence, a result of Hoffman [3, pp. 106-107] asserts that $\hat{b} \circ L_\varphi(z) = \lambda_\varphi z$ for every $z \in \mathbb{D}$, where $\lambda_\varphi \in \mathbb{C}$ has modulus 1. Moreover, if (z_α) is a subnet of $\{z_n\}$ that tends to φ, then $b \circ L_{z_\alpha} \to \hat{b} \circ L_\varphi$ pointwise, and since all the functions $b \circ L_{z_\alpha}$ take real values on the interval $(-1, 1)$, then so does $\hat{b} \circ L_\varphi$. Therefore $\lambda_\varphi = 1$ or -1 (depending on φ). It can be shown that both possibilities occur, but we are not going to use this fact here. The symmetry of Ω_ε with respect to the origin implies that $\hat{b} \circ L_\varphi(z) = \lambda_\varphi z \in \Omega_\varepsilon$ for every $z \in \Omega_\varepsilon$. Consequently $L_\varphi(\Omega_\varepsilon) \subset E$, and since $L_\varphi(\mathbb{D}) \subset \mathcal{M} \setminus \mathbb{D}$, then

$$L_\varphi(\Omega_\varepsilon) \subset E \setminus \mathbb{D} \quad \text{for every} \quad \varphi \in \mathrm{clos}_{\mathcal{M}}\{z_n\} \setminus \mathbb{D}. \tag{3}$$

Since Ω_ε is polynomially convex and $0 \notin \Omega_\varepsilon$, then Runge's theorem asserts that the function $1/\omega$ is the uniform limit on Ω_ε of some sequence of polynomials $\{p_n(\omega)\}$. Therefore $p_n((b(z))$ converges uniformly to $1/b(z)$ on F, showing that $1/b \in A^\infty(F)$.

We are going to prove that $1/b$ cannot be decomposed as in (1) with $p = \infty$. Suppose otherwise that there exist $h \in H^\infty$ and a function f uniformly continuous on F such that $1/b = h + f$ on F. By a simple geometrical argument, the last two items of Lemma 1 and (2) imply that $\mathrm{clos}_{\mathbb{C}}F = F \cup \{1\}$. Since f is uniformly continuous on F, then it can be extended to some continuous function on $F \cup \{1\}$. That is, there is $c \in \mathbb{C}$ such that $\lim f(z) = c$ when $z \to 1$ with $z \in F$. Therefore $1/b(z) - (h(z) + c) \to 0$ when $z \to 1$ with $z \in F$. Multiplying this expression by b and writing $h_1 = h + c$ we obtain

$$1 - b(z)h_1(z) \to 0 \quad \text{when} \quad z \to 1 \quad \text{with} \quad z \in F. \tag{4}$$

Suppose that $x \in E \setminus \mathbb{D}$ and let (z_α) be a net in F that converges to x. It is clear that if we look at (z_α) as a net in the topological space $\mathbb{C}$, then $z_\alpha \to 1$. It follows from (4) that $1 - \hat{b}(x)\hat{h}_1(x) = \lim(1 - b(z_\alpha)h_1(z_\alpha)) = 0$. That is, $1 - \hat{b}\hat{h}_1 \equiv 0$ on $E \setminus \mathbb{D}$. Therefore, for every $\varphi \in \mathrm{clos}_{\mathcal{M}}\{z_n\} \setminus \mathbb{D}$ the inclusion of (3) implies that $1 - \hat{b}\hat{h}_1 \equiv 0$ on $L_\varphi(\Omega_\varepsilon)$. So, writing $g_\varphi = \hat{h}_1 \circ L_\varphi \in H^\infty$, we get

$$0 = 1 - (\hat{b} \circ L_\varphi(z))(\hat{h}_1 \circ L_\varphi(z)) = 1 - \lambda_\varphi z g_\varphi(z) \quad \text{for } z \in \Omega_\varepsilon.$$

Since the only analytic function that vanishes on Ω_ε is the trivial function, then $1 - \lambda_\varphi z g_\varphi(z) = 0$ for every $z \in \mathbb{D}$, which is clearly impossible (take $z = 0$).

3. More general impossibilities

A further analysis of our example with the aid of additional theory will rule out more general decompositions of the same type for $1/b$. The arguments are outlined below.

Suppose that $K \subset \mathcal{M}$ is a closed set and $V \subset \mathcal{M}$ is an open neighborhood of K. Let $h \in H^\infty(V \cap \mathbb{D})$. The corona theorem easily implies that $\operatorname{clos}_{\mathcal{M}}(V \cap \mathbb{D})$ is a neighborhood of K. In addition, h can be continuously extended to some open neighborhood of K in $\mathcal{M}$ (see [5, Thm. 3.2]).

Proposition 2. *Let $\varepsilon(\delta)$, $r(\delta)$ as in Lemma 1 and assume the hypotheses of the example. Then there is no decomposition of the form*

$$1/b = h + f \quad on \quad F, \tag{5}$$

with $f \in C_{ua}(F)$ and $h \in H^\infty(\bigcup K(z_n, r(\delta)))$.

Proof. Suppose that $h \in H^\infty(\bigcup K(z_n, r(\delta)))$ and let η with $\varepsilon < \eta < \varepsilon(\delta)$. The set $V = \{x \in \mathcal{M} : |\hat{b}(x)| < \varepsilon(\delta)\}$ is an open neighborhood of the closed set $K = \{x \in \mathcal{M} : |\hat{b}(x)| \leq \eta\}$. By Lemma 1 $V \cap \mathbb{D} \subset \bigcup K(z_n, r(\delta))$, and consequently the comments preceding the proposition imply that h can be extended to some continuous function on K, say $\tilde{h}$.

Since $|b(\omega)| < \eta$ for every $\omega \in K(z_n, \eta) = L_{z_n}(\Delta(0,\eta))$, then $L_\varphi(\Delta(0,\eta)) \subset K$ for every $\varphi \in \operatorname{clos}_{\mathcal{M}}\{z_n\}$. So, $\tilde{h}$ is continuous on

$$\{L_\varphi(\Delta(0,\eta)) : \varphi \in \operatorname{clos}_{\mathcal{M}}\{z_n\}\}.$$

Let $\varphi \in \operatorname{clos}_{\mathcal{M}}\{z_n\} \setminus \mathbb{D}$ and (z_α) be a subnet of $\{z_n\}$ that tends to φ. The continuity of $\tilde{h}$ on the above set implies that $h \circ L_{z_\alpha}(z)$ tends pointwise to the (necessarily analytic) function $\tilde{h} \circ L_\varphi(z)$ for $|z| < \eta$.

These facts allow us to repeat almost word-by-word the arguments of the previous section. In fact, if (5) holds with $f \in C_{ua}(F)$ and $h \in H^\infty(\bigcup K(z_n, r(\delta)))$, then by the same reasons as in Section 2 there is $c \in \mathbb{C}$ such that the function $g_\varphi(z) = (\tilde{h} + c) \circ L_\varphi(z)$ is analytic on $\Delta(0,\eta)$ and $1 - \lambda_\varphi z g_\varphi(z) = 0$ for $z \in \Omega_\varepsilon$ (observe that $\Omega_\varepsilon \subset \Delta(0,\eta)$). Again, this is impossible.

We recall that the Bloch space B consists of the analytic functions f on $\mathbb{D}$ such that

$$\|f\|_B \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

It is well known (see [6, p. 81]) that every $h \in B$ is uniformly continuous with respect to the metric ρ. An immediate consequence of Proposition 2 is that the decomposition (5) is not possible with $f \in C_{ua}(F)$ and $h \in B$. To see this suppose that there is such decomposition. Then h is bounded on F, and since by Lemma 1 $\rho(z_n, F) < r(\delta) < 1$ for every n, the uniform continuity of h with respect to ρ implies that h is bounded on $\{z_n\}$. By the same reason h must be bounded on $\bigcup K(z_n, r(\delta))$. That is, $h \in H^\infty(\bigcup K(z_n, r(\delta)))$, which contradicts the proposition.

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Departamento de Matemática, Facultad de Cs. Exactas y Naturales, UBA, Pab. I, Ciudad Universitaria, (1428) Núñez, Capital Federal, Argentina

Current address: Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain

E-mail address: `dsuarez@dm.uba.ar`