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ABSTRACT COMPETITIVE SYSTEMS AND ORBITAL STABILITY IN $\mathbb{R}^3$

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ABSTRACT. Competitive autonomous systems in $\mathbb{R}^3$ have the remarkable property of verifying an analogue of the Poincaré-Bendixon theorem for planar equations. This fact allows us to prove the existence of orbitally stable closed orbits for those systems under easily checkable hypothesis. Our aim is to introduce, by changing the ordering in $\mathbb{R}^3$, a new class of autonomous systems for which the preceding results directly extend. As a consequence we shall reinterpret some of the results of R. A. Smith in terms of the theory of monotone systems.

1. INTRODUCTION

In this paper we consider an autonomous differential equation in $\mathbb{R}^3$ and study the existence of stable closed orbits. The papers of R. A. Smith [11] and of H. Zhu and H. L. Smith [13] contain interesting results on this subject and we shall find some connections between them.

Let us consider a system

$$\dot{\xi} = F(\xi), \quad \xi \in \mathbb{R}^3, \tag{1}$$

with $F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. Let us also assume that the system is dissipative and has a unique equilibrium. This equilibrium is hyperbolic and the stable manifold has dimension one. With this simple geometry and some additional conditions on the vector field we would like to find a stable closed orbit.

The work of R. A. Smith [11], also applicable in higher dimensions, gives a first answer to this question. The stable closed orbit exists if one can find a symmetric and nonsingular matrix P, with two negative eigenvalues, and positive constants λ and ϵ such that

$$PF'(\xi) + F'(\xi)^* P + \lambda P \le -\epsilon I \quad \forall \xi \in \mathbb{R}^3, \tag{2}$$

where $\le$ refers to the standard order in the class of symmetric matrices. In the proof of his result in [11], the matrix P is employed to produce an indefinite Lyapunov function that helps to construct the so-called "amenable set". This is an invariant surface that attracts all orbits.

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Another answer to our question can be obtained from the work of H. Zhu and H. L. Smith [13]. A stable closed orbit will exist if the system is competitive and irreducible (see [13] for more details on this definition). The proof of [13] is inspired by a result of M. W. Hirsch (see [3, 4, 5]) that says that a competitive system in $\mathbb{R}^N$ behaves essentially like a system in $\mathbb{R}^{N-1}$. The reduction of dimension plays again an important role.

A system is competitive if the vector field F satisfies

$$\frac{\partial F_i}{\partial \xi_j}(\xi) \leq 0 \quad \text{if} \quad i \neq j, \quad \forall \xi \in \mathbb{R}^3.$$

This is equivalent to saying that the flow $\Phi_t(\xi)$ associated to (1) is monotone in the past; that is,

$$\xi \leq \eta \Rightarrow \Phi_t(\xi) \leq \Phi_t(\eta) \quad \text{if} \quad t < 0.$$

Here we are using the natural ordering in $\mathbb{R}^3$ produced by the cone $\mathbb{R}^3_+$ of vectors with nonnegative components. The results in [13] were stated for competitive systems but they are also valid for flows which are monotone (in the past) with respect to orderings induced by other cones. This is the link between [11] and [13]. With the help of the matrix P we shall construct an ordering such that the monotonicity of a flow is equivalent to saying that the matrix

$$PF'(\xi) + F'(\xi)^* P + \lambda(\xi)P$$

is negative semidefinite. Now $\lambda = \lambda(\xi)$ is a function from $\mathbb{R}^N$ into $\mathbb{R}$. This point of view leads to an extension of the results in [11] for three dimensions. Also, we can interpret the condition of R. A. Smith as a sort of abstract competition.

The rest of the paper is organized as follows. In Section 2 we introduce some terminology on cones and state the main result of [13] in an abstract setting. In Section 3 we define an ordering associated to an indefinite quadratic form and analyze the conditions for the monotonicity of flows with respect to this ordering. Finally, in Section 4 we present a result on orbital stability and discuss the connections with [11].

2. Abstract monotone flows

A closed and convex subset K of $\mathbb{R}^N$ is a cone if it satisfies the properties:

1. If $\xi \in K$, then $\lambda\xi \in K$ for each $\lambda \geq 0$.
2. $K \cap (-K) = \{0\}$.

Each cone induces an ordering in $\mathbb{R}^N$ defined by

$$\xi \leq \eta \quad \text{if and only if} \quad \eta - \xi \in K.$$

Of special importance are cones with nonempty interior. Whenever $\mathring{K} \neq \emptyset$, the following notation is employed:

$$\xi \ll \eta \quad \text{if and only if} \quad \eta - \xi \in \mathring{K}.$$

The notation $\xi < \eta$ just means $\xi \leq \eta$ and $\xi \neq \eta$. We shall discuss some connections between cones and differential equations. Given the system

$$\dot{\xi} = F(\xi), \quad \xi \in \mathbb{R}^N, \tag{3}$$

with $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, the solution starting at ξ is denoted by $\Phi_t(\xi)$ and the corresponding maximal interval is I_ξ. Also, $I_\xi^- = I_\xi \cap (-\infty, 0)$, $I_\xi^+ = I_\xi \cap (0, +\infty)$.

The flow associated to (3) is said to be monotone in the past (resp. in the future) if, given $\xi, \eta \in \mathbb{R}^N$ with $\xi < \eta$, then $\Phi_t(\xi) < \Phi_t(\eta)$ for each $t \in I_\xi^- \cap I_\eta^-$ (resp. $t \in I_\xi^+ \cap I_\eta^+$).

As mentioned in the introduction, M. W. Hirsch considered the cone $K = \mathbb{R}_+^N$ and proved that the study of the dynamics of a monotone system in $\mathbb{R}^N$ can be reduced in one dimension. In particular, the conclusion of the Poincaré-Bendixon theorem still holds for monotone flows in $\mathbb{R}^3$ (see [3, 5]). All this theory can be extended to flows that are monotone with respect to the ordering induced by a cone K with $\overset{\circ}{K} \neq \emptyset$. The crucial property of these orderings is that intervals of the kind

$$]\xi, \eta[= \{\theta \in \mathbb{R}^N : \xi \ll \theta \ll \eta\}$$

are open and bounded in $\mathbb{R}^N$ (see [7, pages 37–38]).

Let $D\Phi_t(\xi)$ denote the derivative of the flow with respect to initial conditions. This is the solution matrix of the linear problem

$$\dot{X}(t) = F'(\Phi_t(\xi))X(t), \quad X(0) = I_N.$$

We shall say that the flow associated to (3) is strongly L-monotone (in the past) if for each $\xi \in \mathbb{R}^N$ and $\delta \in K - \{0\}$ one has

$$D\Phi_t(\xi)\delta \gg 0 \quad \forall t \in I_\xi^- . \tag{4}$$

Let us see that a flow of this kind is monotone. Actually, given $\xi, \eta \in \mathbb{R}^N$ with $\xi < \eta$ and t negative and sufficiently small,

$$\Phi_t(\eta) - \Phi_t(\xi) = \int_0^1 D\Phi_t(\tau\eta + (1-\tau)\xi)(\eta - \xi)d\tau$$

and it follows from (4) that this difference is positive. The reader will notice that we have used the monotonicity of the vector valued integral with respect to the order induced by K. Actually, given any continuous function $f : [0,1] \to \overset{\circ}{K}$, then the average $\int_0^1 f(\tau)d\tau$ lies also in $\overset{\circ}{K}$. An easy way to prove this fact is as follows. Consider the compact set $C = f([0,1]) \subset \overset{\circ}{K}$. The convexity of $\overset{\circ}{K}$ implies that the convex hull $\operatorname{co} C$ is contained in $\overset{\circ}{K}$. We know that $\operatorname{co} C$ is compact (see [8, page 72]) and the mean value theorem for vector-valued integrals (see [8, page 75]) leads to the conclusion.

Going back to the proof of the monotonicity of the flow Φ_t, we already know that, for some small $\epsilon > 0$,

$$\Phi_t(\xi) < \Phi_t(\eta) \quad \text{if} \quad t \in (-\epsilon, 0).$$

To prove that this inequality also holds for all $t \in I_\xi^- \cap I_\eta^-$, it is sufficient to notice that $I_\xi^- \cap I_\eta^- \subset I_{\tau\xi+(1-\tau)\eta}^-$. This is a consequence of the local monotonicity already proved.

The following theorem is a partial extension of the main result of [13] to general cones. We recall that the system (3) is said to be dissipative if there exists a compact set $D \subset \mathbb{R}^N$ such that, for all $\xi \in \mathbb{R}^N$, $\Phi_t(\xi)$ belongs to D for all t sufficiently large.

Theorem 1. *Let K be a cone in $\mathbb{R}^3$ with $\overset{\circ}{K} \neq \emptyset$. Let us suppose that system* (1) *is dissipative, its flow is strongly L-monotone in the past (with respect to the ordering* *induced by K) and there exists a unique equilibrium $\xi = 0$ such that the eigenvalues of $F'(0)$ satisfy*

$$\lambda_1 < 0, \, \mathrm{Re}\, \lambda_2, \mathrm{Re}\, \lambda_3 > 0.$$

Then there exists at least one orbitally stable closed orbit. Moreover, every orbit tends to the equilibrium or to a closed orbit as $t \to +\infty$.

Proof. The existence of a stable closed orbit is proven essentially in the same way as in [13]. To prove the convergence of every orbit to $\xi = 0$ or to a closed orbit we use Theorem 4.2 of Chapter 3 in [9]. Again this result is stated for $K = \mathbb{R}^3_+$ but extends to general cones. We must prove that there is a vector in $\overset{\circ}{K}$ that is tangent to the stable manifold $W^s(0)$ at $\xi = 0$. The matrix solution of the linearized equation $\dot{\xi} = F'(0)\xi$ is $\exp(F'(0)t) = D\Phi_t(0)$. By assumption this matrix satisfies

$$D\Phi_t(0)(K - \{0\}) \subset \overset{\circ}{K} \text{ for } t < 0.$$

The Perron-Frobenius theorem for abstract cones (see [1, 7]) implies that the eigenvector of $\exp(F'(0)t)$ associated to $\exp(\lambda_1 t)$ belongs to $\overset{\circ}{K}$. This vector is tangent to $W^s(0)$ at $\xi = 0$. □

Under the assumptions of the previous theorem one can also prove the existence of an asymptotically stable closed orbit when all closed orbits are isolated. This is the case if F is analytic in the real sense.

3. Quadratic cones and Kamke conditions

In this section the dimension will not play a relevant role and we shall always work in $\mathbb{R}^N$ with arbitrary $N \geq 2$. We shall fix a symmetric matrix P of dimension N having one positive eigenvalue and $N-1$ negative eigenvalues (each eigenvalue is counted according to its multiplicity). The positive eigenvalue will be denoted by λ_+ and e_+ will stand for an eigenvector satisfying

$$Pe_+ = \lambda_+ e_+, \; |e_+| = 1.$$

We define the set

$$K = \{\xi \in \mathbb{R}^N : (P\xi, \xi) \geq 0, \, (\xi, e_+) \geq 0\}$$

where (,) is the usual inner product in $\mathbb{R}^N$. It is not difficult to prove that K is a cone with nonempty interior. In this proof one uses the Cauchy-Schwartz inequality and the following property:

$$(P\xi, \xi) \geq 0, \; \xi \neq 0 \Rightarrow (\xi, e_+) \neq 0. \tag{5}$$

This is true because P is negative definite on the orthogonal complement of e_+.

From now on the notations $\leq$, $\ll$ and $<$ will always refer to the ordering induced by K. Namely, given $\xi \in \mathbb{R}^N$,

$$\xi \geq 0 \Leftrightarrow (P\xi, \xi) \geq 0 \text{ and } (\xi, e_+) \geq 0,$$

$$\xi \gg 0 \Leftrightarrow (P\xi, \xi) > 0 \text{ and } (\xi, e_+) > 0.$$

We go back to the differential equation (3). We shall say that (3) is P-competitive if there exists a function $\lambda : \mathbb{R}^N \to \mathbb{R}$ such that the symmetric matrix

$$PF'(\xi) + F'(\xi)^* P + \lambda(\xi) P$$

is negative semidefinite for each $\xi \in \mathbb{R}^N$. If this matrix is negative definite everywhere, we shall say that the system is strictly P-competitive.

Theorem 2. *The flow generated by* (3) *is monotone in the past (with respect to* K*) if and only if* (3) *is* P*-competitive. Moreover, if* (3) *is strictly* P*-competitive, then the flow is strongly* L*-monotone.*

Remark 1. In his works [10, 11], R. A. Smith assumed that the vector field F was locally Lipschitz continuous and satisfied the condition

$$(\xi - \eta, P(F(\xi) - F(\eta)) + \lambda P(\xi - \eta)) \leq -\epsilon|\xi - \eta|^2 \tag{6}$$

where ϵ and λ are positive constants. When one assumes that F is of class C^1, this condition is equivalent to (2). The proof of the implication (6) $\Rightarrow$ (2) can be seen in the proof of Lemma 5 in [12], and the converse is a consequence of the identity

$$(\xi - \eta, P(F(\xi) - F(\eta))) = \int_0^1 (PF'(\tau\xi + (1-\tau)\eta)(\xi - \eta), \xi - \eta)d\tau.$$

Thus the condition (6) implies that the system is P-competitive.

Remark 2. From Theorem 2, the previous remark and the results in [3, 5, 9] we obtain a partial generalization of the Extended Poincaré-Bendixon Theorem of R. A. Smith [10, 11] in the case $N = 3$. Namely, in a three-dimensional P-competitive system, compact omega limit sets without equilibrium points are closed orbits. Our extension of R. A. Smith's result is only partial because we do not consider the case of locally Lipschitz continuous fields and also because in [10, 11] the field F is not necessarily defined in the whole space $\mathbb{R}^3$. On the other hand, P-competitivity is a condition more flexible than (2). In R. A. Smith's work the matrix

$$PF'(\xi) + F'(\xi)^*P + \lambda P$$

is negative definite in a uniform sense with respect to ξ, and λ is a constant. These two requirements are not needed for strict P-competitivity. At the end of the paper we show an example emphasizing this point.

Remark 3. It is interesting to analyze the class of P-competitive systems for some simple matrices. For example, if $N = 2$ and

$$P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

then (3) is P-competitive if and only if

$$\left| \frac{\partial F_1}{\partial \xi_2} - \frac{\partial F_2}{\partial \xi_1} \right| \leq \frac{\partial F_1}{\partial \xi_1} - \frac{\partial F_2}{\partial \xi_2}.$$

Remark 4. If we suppose that F has a bounded derivative in $\mathbb{R}^N$, then flow associated to (3) can be embedded in a monotone flow of $\mathbb{R}^{N+1}$. To be more specific, it is not difficult to see that the system

$$\begin{cases} \dot{\xi} &= F(\xi), \quad \xi \in \mathbb{R}^N, \\ \dot{\eta} &= -\alpha\,\eta, \quad \eta \in \mathbb{R}, \end{cases}$$

is strictly P-competitive if $\alpha \in \mathbb{R}$ is great enough, where P is the matrix

$$P = \begin{pmatrix} -I_n & 0 \\ 0 & 1 \end{pmatrix}.$$

Before the proof of Theorem 2 we need the following algebraic result.

Lemma 1. *Let A be a symmetric matrix of dimension N. The following statements are equivalent:*

(i) *For each $\xi \in \mathbb{R}^N$, $\xi \neq 0$, satisfying $(P\xi, \xi) = 0$, the inequality below holds:*

$$(A\xi, \xi) < 0.$$

(ii) *There exists $\lambda \in \mathbb{R}$ such that $A + \lambda P$ is negative definite.*

Proof. The implication (ii) $\Rightarrow$ (i) is immediate. To prove the converse let us assume that (i) holds.

Define the sets

$$H_+ = \{\xi \in \mathbb{R}^N : (P\xi, \xi) = 1\} \quad \text{and} \quad H_- = \{\xi \in \mathbb{R}^N : (P\xi, \xi) = -1\}.$$

Let us prove first that

$$(A\xi, \xi) \to -\infty \text{ as } |\xi| \to +\infty, \ \xi \in H_+ \cup H_-. \tag{7}$$

If not, there would exist $M > 0$ and $\xi_n \in H_+ \cup H_-$ such that

$$\lim_{n\to+\infty} |\xi_n| = +\infty \quad \text{and} \quad (A\xi_n, \xi_n) \geq -M.$$

Let us call $\theta_n = \frac{\xi_n}{|\xi_n|}$. Then $\{\theta_n\}$ is a sequence of unitary vectors verifying

$$(A\theta_n, \theta_n) \geq \frac{-M}{|\xi_n|^2} \quad \text{and} \quad (P\theta_n, \theta_n) = \frac{c}{|\xi_n|^2} \tag{8}$$

where $c = \pm 1$.

Passing to a subsequence if necessary, we can suppose that $\{\theta_n\}$ converges to a nonzero vector θ_0. Taking limits in (8) we have that

$$(A\theta_0, \theta_0) \geq 0 \quad \text{and} \quad (P\theta_0, \theta_0) = 0$$

which contradicts (i).

Property (7) allows us to define the numbers

$$\mu_+ = \max\{(A\xi, \xi) : \xi \in H_+\}$$

and

$$\mu_- = \max\{(A\xi, \xi) : \xi \in H_-\}.$$

It follows that

$$(A\xi, \xi) \leq \mu_+(P\xi, \xi) \quad \text{if} \quad (P\xi, \xi) > 0 \tag{9}$$

and

$$(A\xi, \xi) \leq -\mu_-(P\xi, \xi) \quad \text{if} \quad (P\xi, \xi) < 0. \tag{10}$$

Let $v_+ \in H_+$ and $v_- \in H_-$ be vectors where the maxima are reached. The Lagrange multipliers theorem implies that they satisfy

$$Av_+ = \mu_+ P v_+, \quad Av_- = -\mu_- P v_-. \tag{11}$$

Let us suppose for a while that $\mu_+ = -\mu_- = \mu$ and let Π be the plane generated by v_+ and v_-. Then we would have that

$$A\xi = \mu P\xi \quad \forall \xi \in \Pi.$$

Since $(Pv_+, v_+) > 0$ and $(Pv_-, v_-) < 0$, there would exist $v \in \Pi$, $v \neq 0$, satisfying $(Pv, v) = 0$. Then

$$(Av, v) = \mu(Pv, v) = 0$$

which contradicts (i). So $\mu_+ \neq -\mu_-$. This fact and (11) easily imply that

$$(Av_+, v_-) = (Pv_+, v_-) = 0.$$

From the properties of v_- and v_+ we deduce that the vector $v_0 = v_+ + v_-$ satisfies $(Pv_0, v_0) = 0$. Applying (i) to v_0 we obtain that

$$(Av_0, v_0) = \mu_+ + \mu_- < 0.$$

Taking $\lambda \in \mathbb{R}$ such that

$$\mu_- < \lambda < -\mu_+ \tag{12}$$

let us prove that

$$(A\xi, \xi) + \lambda(P\xi, \xi) < 0 \quad \forall \xi \neq 0.$$

The case $(P\xi, \xi) = 0$ is straightforward from (i).

If $(P\xi, \xi) > 0$, then (9) and (12) imply

$$(A\xi, \xi) + \lambda(P\xi, \xi) \leq (\mu_+ + \lambda)(P\xi, \xi) < 0.$$

If $(P\xi, \xi) < 0$, then (10) and (12) imply

$$(A\xi, \xi) + \lambda(P\xi, \xi) \leq (-\mu_- + \lambda)(P\xi, \xi) < 0.$$

This completes the proof. □

Proof of Theorem 2. First we assume that the system is strictly P-competitive and we shall prove that the flow is strongly L-monotone. Given $\delta > 0$ we must prove that

$$D\Phi_t(\xi)\delta \gg 0 \text{ if } t \in I_\xi^-.$$

The function $y(t) = D\Phi_t(\xi)\delta$ is the solution of

$$\dot{y} = F'(\Phi_t(\xi))y, \quad y(0) = \delta.$$

Define $a_1(t) = (Py(t), y(t))$. The derivative satisfies

$$\dot{a_1}(t) = ((PF'(\Phi_t(\xi)) + F'(\Phi_t(\xi))^* P)y(t), y(t)) < -\lambda(\Phi_t(\xi))a_1(t).$$

In particular we deduce that $\dot{a_1}(t_0) < 0$ if $a_1(t_0) = 0$. Since $\delta > 0$, we know that $a_1(0) \geq 0$ and therefore a_1 must be positive for negative t. Next we define $a_2(t) = (y(t), e_+)$. It follows from (5) that a_2 cannot vanish when $a_1(t)$ is positive. From $a_2(0) > 0$ we deduce that also a_2 is positive in I_ξ^-. This proves that $y(t) \gg 0$ if $t \in I_\xi^-$.

Now we assume that the system is P-competitive and we prove that the flow is monotone. This follows easily by an approximation argument. For each $\epsilon > 0$ the system

$$\dot{\xi} = F(\xi) - \epsilon P^{-1}\xi$$

is strictly P-competitive. In consequence the associated flow is strongly L-monotone and so it is monotone. For $\epsilon = 0$ the flow is also monotone by continuous dependence.

To finish the proof we assume that (3) is not P-competitive and we prove that the flow is not monotone in the past. If the system is not P-competitive, we apply the preceding lemma with $A = PF'(\xi) + F'(\xi)^* P$ to deduce the existence of ξ_0 and η in $\mathbb{R}^N$ such that

$$(P\eta, \eta) = 0 \text{ and } (PF'(\xi_0)\eta, \eta) = \frac{1}{2}((PF'(\xi_0) + F'(\xi_0)^* P)\eta, \eta) > 0.$$

Notice that the equivalence in that lemma still holds when the inequality in (i) is not strict and $A + \lambda P$ is negative semidefinite.

We can choose η so that $(\eta, e_+) > 0$. This follows from (5) and implies $\eta > 0$. Let $\epsilon > 0$ be such that

$$(P(F(\xi_0 + \epsilon\eta) - F(\xi_0)), \eta) > 0.$$

This is precisely the derivative at $t = 0$ of

$$a_3(t) = \frac{1}{2\epsilon}(P(\Phi_t(\xi_0 + \epsilon\eta) - \Phi_t(\xi_0)), \Phi_t(\xi_0 + \epsilon\eta) - \Phi_t(\xi_0)).$$

Since $a_3(0) = 0$, the inequality $\Phi_t(\xi_0 + \epsilon\eta) > \Phi_t(\xi_0)$ cannot hold for small negative values of t. This shows that the flow is not monotone in the past. □

4. A result on orbital stability

Theorem 1 together with the previous result lead to the following result on orbital stability.

Theorem 3. *Let us suppose that system* (1) *is dissipative, strictly P-competitive and there exists a unique equilibrium $\xi = 0$ such that the eigenvalues of $F'(0)$ satisfy*

$$\lambda_1 < 0, \text{Re } \lambda_2, \text{Re } \lambda_3 > 0.$$

Then the conclusions of Theorem 1 hold.

Remark 5. The assumption of hyperbolicity of $\xi = 0$ can probably be relaxed but not completely removed. For example, we can consider the planar autonomous system appearing in [2, pages 191–194]. This is a dissipative system having a unique equilibrium, which is unstable, and having no closed orbit. Multiplying the vector field by a suitable scalar function we obtain another equation with the same phase portrait and such that $F(\xi)$ has bounded derivative. We can now use Remark 4 to construct a three dimensional system that is strictly P-competitive, dissipative and has a unique unstable equilibrium and no closed orbit.

Remark 6. The previous theorem extends the results in [11] if $N = 3$. However it cannot be considered as a complete extension of the result of R. A. Smith due to the reasons already discussed in Remark 2.

To conclude the paper we give an example for which the assumption of R. A. Smith in [11] does not hold while Theorem 3 is applicable. Let us consider the system

$$\begin{aligned} \dot{x} &= x - y - xz^2 - \frac{x(x^2+y^2)}{3}, \\ \dot{y} &= x + y - yz^2 - \frac{y(x^2+y^2)}{3}, \\ \dot{z} &= -z - \frac{z^3}{3} - z(x^2+y^2). \end{aligned} \tag{13}$$

Using the function $V(x,y,z) = x^2 + y^2 + z^2$ it is easy to see that this system is dissipative. The linearization in the origin, which is its unique equilibrium point, has as eigenvalues -1, $1+i$ and $1-i$. If we take

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a computation shows that (13) is strictly P-competitive, where λ must satisfy

$$2(x^2+y^2+z^2-1) < \lambda(x,y,z) < 2(x^2+y^2+z^2+1).$$

So λ cannot be constant.

Finally, it can be seen that $x(t) = \sqrt{3}\cos t$, $y(t) = \sqrt{3}\sin t$, $z(t) = 0$ is an orbitally stable solution of (13).

Final remark. After we finished this paper we learned from E. N. Dancer that the quadratic cones had been already employed by Hofer and Toland in [6] to study a class of hamiltonian systems.

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