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# CORRECTION TO "THE GHOST OF AN INDEX THEOREM"

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ABSTRACT. The "ghost of an index theorem" is an isomorphism between products of the kernel spaces and the cokernel spaces of a pair of bounded operators and their product, valid when each operator and also the product is assumed to have a generalized inverse. In this note we correct an error in the original proof, and extend the result to operators with closed range.

Suppose $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators between Banach spaces, with *generalized inverses* $T' : Y \to X$ and $S' : Z \to Y$:

$$T = TT'T, \quad S = SS'S; \tag{0.1}$$

suppose also that the product $ST : X \to Y$ has a generalized inverse $U : Z \to X$:

$$ST = STUST. \tag{0.2}$$

Under these assumptions we have claimed

**1. Theorem.** *There is an isomorphism*

$$T^{-1}(0) \times S^{-1}(0) \times Z/ST(X) \cong (ST)^{-1}(0) \times Y/T(X) \times Z/S(Y). \tag{1.1}$$

In the proof ([1, Theorem 2]) we constructed matrices

$$\mathbf{U} = \begin{pmatrix} I - T'T & 0 & 0 \\ (I - TUS)T & I - TUS & 0 \\ 0 & S(I - TUS) & I - SS' \end{pmatrix} \tag{1.2}$$

and

$$\mathbf{V} = \begin{pmatrix} I - T'T & T'(I - S'S) & 0 \\ 0 & (I - TT')(I - S'S) & (I - TT')S' \\ 0 & 0 & I - SS' \end{pmatrix}, \tag{1.3}$$

having first normalized the generalized inverses of (0.1) and (0.2) so that

$$T' = T'TT', \quad S' = S'SS', \quad U = USTU, \quad U = T'VS'. \tag{1.4}$$

The argument then proceeds by claiming that $\mathbf{U}$ and $\mathbf{V}$ are mutually generalized inverse, and identifying their ranges with the products of (1.1). Unfortunately $\mathbf{U}$ and $\mathbf{V}$ may fail to be mutually generalized inverse, and their products need not be diagonal as claimed in [1]. To see how to modify their definition we must go back to the source, which is the "one diagram" proof of the index theorem due to Yang [4].

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The argument of Yang ([4, Theorem]) proceeds by the construction of an exact sequence of bounded operators

(1.5)
$$0 \overset{0}{\longleftarrow} W_5 \overset{A_5}{\longleftarrow} W_4 \overset{A_4}{\longleftarrow} W_3 \overset{A_3}{\longleftarrow} W_2 \overset{A_2}{\longleftarrow} W_1 \overset{A_1}{\longleftarrow} W_0 \overset{0}{\longleftarrow} 0$$

where

$$W_0 = T^{-1}(0), \quad W_1 = (ST)^{-1}(0), \quad W_2 = S^{-1}(0), \qquad (1.6)$$
$$W_3 = Y/T(X), \quad W_4 = Z/(ST)(X), \quad W_5 = Z/S(Y),$$

and then the mappings $A_j$ are natural: for example, $A_1$ is an inclusion map and $A_5$ is a quotient; $A_2$ and $A_4$ are induced by $T$ and $S$, respectively, and $A_3$ comes from the "snake lemma". Our strategy in [1] is to replace the kernel and cokernel spaces by the originals:

$$W_0 = W_1 = X, \quad W_2 = W_3 = Y, \quad W_4 = W_5 = Z\ . \qquad (1.7)$$

The mappings $A_j$ are now more complicated, in that we must define them compatibly on the whole space rather than only on a subspace or a quotient: for example, $A_1$ and $A_5$ are the projections induced by the generalized inverses of $T$ and $S$. Specifically

$$A_1 = I - T'T, \quad A_2 = (I - TUS)T, \quad A_3 = (I - TT')(I - S'S), \qquad (1.8)$$
$$A_4 = S(I - TUS), \quad A_5 = I - SS'.$$

Taking account of the normalization (1.4) we now have an example of a *regular chain*, in the sense that (with a little *abus de notation*)

$$A_5A_4 = A_4A_3 = A_3A_2 = A_2A_1 = 0, \qquad (1.9)$$

and each $A_j$ has a generalized inverse $A_j'$:

$$A_1' = I - T'T, \quad A_2' = T'(I - S'S), \quad A_3' = I - TUS, \qquad (1.10)$$
$$A_4' = (I - TT')S, \quad A_5' = I - SS'.$$

The behaviour of such a regular chain is described ([3, (2.5), (2.6)]) by the matrices

$$\mathbf{U} = \begin{pmatrix} A_1' & 0 & 0 \\ A_2 & A_3' & 0 \\ 0 & A_4 & A_5' \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} A_1 & A_2' & 0 \\ 0 & A_3 & A_4' \\ 0 & 0 & A_5 \end{pmatrix}. \qquad (1.11)$$

We can now seek to show that $\mathbf{U}$ and $\mathbf{V}$ are mutually generalized inverse, and identify their ranges; but for this to work it is crucial that

(1.12)
$$0 \overset{0}{\longleftarrow} W_0 \overset{A_1'}{\longleftarrow} W_1 \overset{A_2'}{\longleftarrow} W_2 \overset{A_3'}{\longleftarrow} W_3 \overset{A_4'}{\longleftarrow} W_4 \overset{A_5'}{\longleftarrow} W_5 \overset{0}{\longleftarrow} 0$$

is another chain; we need

$$A_1'A_2' = A_2'A_3' = A_3'A_4' = A_4'A_5' = 0. \qquad (1.13)$$

This need not happen; however it is rather easy ([3, page 284]) to replace the generalized inverses $A_j'$ by generalized inverses $A_j''$ which form a chain:

**2. Lemma.** *If $T = TT'T$ and $S = SS'S$ with $ST = 0$, then*

$$T = TT''T, \quad S = SS''S, \quad T''S'' = 0 \tag{2.1}$$

*where*

$$(T'', S'') = (T', (I - TT')S') \text{ or } (T'', S'') = (T'(I - S'S), S'). \tag{2.2}$$

*Proof.* Clear. •

The extension of this procedure from regular chains of length 2 to regular chains of length $n$ is clear: for the chain $A$ of (1.5) we may take

$$\begin{aligned} A_1'' = A_1', \quad A_2'' = (I - A_1A_1')A_2', \quad A_3'' = (I - A_2A_2'')A_3', \\ A_4'' = (I - A_3A_3'')A_4', \quad A_5'' = (I - A_4A_4'')A_5' \ . \end{aligned} \tag{2.3}$$

If we now write

$$\mathbf{U}' = \begin{pmatrix} A_1'' & 0 & 0 \\ A_2 & A_3'' & 0 \\ 0 & A_4 & A_5'' \end{pmatrix} \quad \text{and} \quad \mathbf{V}' = \begin{pmatrix} A_1 & A_2'' & 0 \\ 0 & A_3 & A_4'' \\ 0 & 0 & A_5 \end{pmatrix}, \tag{2.4}$$

then we find (as claimed spuriously in [1]) that $\mathbf{U}'\mathbf{V}'$ and $\mathbf{V}'\mathbf{U}'$ are each diagonal, with $\mathbf{U}'$ and $\mathbf{V}'$ mutually generalized inverse. Thus to salvage our proof of the ghost of an index theorem [1] we must replace $\mathbf{U}$ and $\mathbf{V}$ of (1.2) and (1.3) by the appropriate $\mathbf{U}'$ and $\mathbf{V}'$ specialized to the $A_j$ and $A_j'$ of (1.7) and (1.9); we leave it to the reader to work this out explicitly.

If we relax the assumptions on the operators $S$, $T$ and $ST$, asking only that all three operators have closed range, then we still have a ghost; we must however work with the operators of Yang as in (1.5):

**3. Theorem.** *If $T : X \to Y$, $S : Y \to Z$ and $ST : X \to Z$ have closed range, then there exist isomorphisms*

$$Z/SY \cong (Z/STX)/(SY/STX), \tag{3.1}$$

$$SY/STX \cong (Y/STX)/((SY + T^{-1}(0))/TX), \tag{3.2}$$

$$(SY + T^{-1}(0))/TX \cong S^{-1}(0)/(S^{-1}(0) \cap TX), \tag{3.3}$$

$$S^{-1}(0) \cap TX \cong (ST)^{-1}(0)/T^{-1}(0). \tag{3.4}$$

*Proof.* The sequence (1.5) is exact at each point $W_j$, and hence for each $j$ there is an isomorphism

$$A_{j+1}(W_j) \cong W_j/A_j(W_{j-1}). \bullet \tag{3.5}$$

If we abandon all assumptions about the operators $S$ and $T$ we might expect to replace ranges by their closure, and hence salvage something; this [2] is however a forlorn hope. The reader is invited to write down the sequence (1.5) when ([2, Example 1]) $T : X \to Y$ is one-one with dense range and $S : Y \to Z$ is of rank one with $S^{-1}(0) \cap TX = \{0\}$.

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