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DIFFERENTIABLE MAPPINGS WITH AN INFINITE NUMBER OF CRITICAL POINTS

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Abstract. In this paper we shall give some sufficient conditions in order that the so-called φ-category of a pair (M, N) of differentiable manifolds be infinite.

Introduction

Let $M,\ N$ be compact connected differentiable (smooth) manifolds having the same dimension $m \geq 2$. If $\pi_q(M) \ncong \pi_q(N)$ for some $q \geq 2$, then any differentiable (smooth) mapping $f : M \to N$ has one critical point at least. Indeed, otherwise f is a covering map and $\pi_q(M) \cong \pi_q(N)$, for $q \geq 2$, follows easily.

This observation justifies the investigations on the cardinal number

$$\varphi(M, N) = \min\{|C(f)| \mid f \in C^\infty(M, N)\}$$

called the φ-category of the pair (M, N), where $M,\ N$ are, at this time, arbitrary differentiable manifolds and $C(f)$ is the critical set of $f \in C^\infty(M, N)$. If any differentiable mapping from M to N has infinitely many critical points, we shall use the notation $\varphi(M, N) = \infty$. As we have already mentioned in our previous papers [AnPi] and [Pi1], $\varphi(M, N)$ represents a measure of non-immersability of M into N, if $dim\, M < dim\, N$ and it is a measure of the distance of the pair (M, N) from a fibration if $dim\, M \geq dim\, N$ and $M,\ N$ are compact differentiable manifolds. Some results concerning the φ-category of the pair $(M, \mathbf{R})$ are obtained in [GoGo] and [Ta]. For the equivariant (invariant) situation see also [Da].

The main result of this paper is Theorem 3.1 which asserts that, for two given compact connected differentiable manifolds M and N having the same dimension m, $\varphi(M, N)$ is infinite in the following conditions:

(i) $m \geq 3$ and $\pi_1(N)$ has no subgroup isomorphic with $\pi_1(M)$;

(ii) $m \geq 4$ and $\pi_q(M) \ncong \pi_q(N)$ for some $q \in \{2, 3, \ldots, m-2\}$.

The proof of Theorem 3.1 is essentially based on the technical Theorems 1.1, 1.2 and on Proposition 2.3 and it is very easy to cover. Proposition 2.3 is an immediate consequence of Theorem 2.2 and the last one could be also considered an important result in itself. Theorem 2.2 roughly asserts that for two given manifolds P and M such that $dim\, P < dim\, M$, $\partial P \neq \emptyset$, $\partial M = \emptyset$ and a discrete subset A of M whose derived set A' (i.e. the set of accumulation points) is also discrete, in each

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homotopy class $[f] \in [(P, \partial P), (M, M\backslash\bar{A})]$ there is a map whose image avoids the set $\bar{A}$. For its proof f is assumed to be smooth and around the common points of the sets $\bar{A}$ and $Im\, f$ we take small balls such that they don't meet $\bar{A}$ again. Using Sard's theorem we can consider in each such ball a point which doesn't belong to $Im\, f$. Further on, project the common points of $Im\, f$ and a given ball on the border of that ball along some certain curves starting from the considered point of the ball and passing through their common points. The asserted mapping in the class $[f]$ is obtained by composing those projections with the initial mapping f.

1. Preliminary results

In their paper [FaHu], E. Fadell and S. Husseini proved, using the standard technique of Ljusternik-Schnirelmann category, that a C^1-functional $f : \Lambda \to \mathbf{R}$ has an unbounded sequence of critical values (therefore infinitely many critical points), under certain hypothesis on the open subset Λ of the Hilbert space E and on the functional f. Our first result proves a similar fact for a mapping between two connected differentiable manifolds.

As we have already defined before, $C(f)$ denotes the critical set of the mapping $f : M \to N$. Recall that $R(f)$ denotes the regular set of f and $B(f) = f(C(f))$ denotes the set of critical values of f.

Theorem 1.1. *Let M , N be connected differentiable manifolds such that $dim\, M \geq dim\, N \geq 2$. If $f : M \to N$ is a non-surjective closed differentiable mapping, then either $C(f) = M$, or f has infinitely many critical values. Therefore f has infinitely many critical points. If M is compact and N is noncompact, then one particularly gets that $\varphi(M, N) = \infty$.*

Proof. Let us first prove that $f^{-1}(\partial Im\, f) \subseteq C(f)$, which implies that $\partial Im\, f \subseteq B(f)$. Indeed, otherwise $f^{-1}(\partial Im\, f) \cap R(f) \neq \emptyset$ and f is locally open around any point of the set $f^{-1}(\partial Im\, f) \cap R(f)$. If $x \in f^{-1}(\partial Im\, f) \cap R(f)$ is a fixed point and U is an open neighbourhood of x such that $f|_U : U \to N$ is open, then $f(U)$ is particularly open. But this is a contradiction to the fact that $f(x) \in \partial Im\, f$.

If $C(f) \neq M$, it follows, by Sard's theorem, that $Im\, f\backslash B(f) \neq \emptyset$. In the following we shall show that $N\backslash B(f)$ is not connected, which means that $B(f)$ is infinite. Indeed, if $y \in Im\, f\backslash B(f)$, $y' \in N\backslash Im\, f$, then obviously y, $y' \in N\backslash B(f)$. Consider $\gamma : [0, 1] \to N$ a continuous path joining y to y'. Because $y \in Im\, f$ and $y' \in N\backslash Im\, f$, it follows that $\gamma([0, 1])$ intersects the border $\partial Im\, f$ and hence the set $B(f)$. □

Theorem 1.2. *Let M, N be compact connected differentiable manifolds having the same dimension $m \geq 2$. If the differentiable mapping $f : M \to N$ is surjective and has finitely many critical points, then the set $f^{-1}(B(f))$ is the closure of a discrete set whose derived set is finite, and the restriction*

$$f|_{M\backslash f^{-1}(B(f))} : M\backslash f^{-1}(B(f)) \to N\backslash B(f)$$

is a covering mapping with finitely many sheets.

Proof. Before proving the required statement we shall prove the inclusion

$$[f^{-1}(B(f)) \cap R(f)]' \subseteq C(f)$$

where $[f^{-1}(B(f)) \cap R(f)]'$ is the derived set of $[f^{-1}(B(f)) \cap R(f)]$. Indeed, if the above inclusion is not true, then there exists a critical value $y \in B(f)$ whose fiber

$f^{-1}(y)$ has a regular point x as accumulation point. But this is a contradiction to the fact that f is a local diffeomorphism around of x. Further on, we have successively:

$$f^{-1}(B(f)) = \overline{f^{-1}(B(f)) \cap M} = \overline{f^{-1}(B(f)) \cap [C(f) \cup R(f)]}$$

$$= \overline{(f^{-1}(B(f)) \cap C(f)) \cup [f^{-1}(B(f)) \cap R(f)]} = \overline{C(f) \cup [f^{-1}(B(f)) \cap R(f)]}.$$

Obviously

$$C(f) \cup [f^{-1}(B(f)) \cap R(f)]$$

is a discrete set whose derived set $[f^{-1}(B(f)) \cap R(f)]'$ is finite, being a subset of $C(f)$, and $f|_{M \backslash f^{-1}(B(f))}$ is a local diffeomorphism. Therefore to prove what is required, it is enough to show that $|f^{-1}(y)| = |f^{-1}(y')| < \infty$ for all $y,\ y' \in N \backslash B(f)$. Let us first see that $f^{-1}(y)$ is finite for all $y \in N \backslash B(f)$. Indeed, otherwise $f^{-1}(y)$ has accumulation points obviously contained in $f^{-1}(y)$. But this contradicts the fact that f is locally a diffeomorphism around any point $x \in f^{-1}(y)$. It remains to prove that the number $|f^{-1}(y)|$ is invariant when y runs over the set $N \backslash B(f)$. We first prove this statement locally, that is, every $y \in N \backslash B(f)$ has an open neighbourhood V_y such that $|f^{-1}(y)| = |f^{-1}(y')| < \infty$ for all $y' \in V_y$. Assume that $f^{-1}(y) = \{x_1, \dots, x_p\}$. Consider $U_1, U_2, \dots, U_p$ open neighbourhoods of $x_1, x_2, \dots, x_p$ respectively, and V an open neighbourhood of y such that $f(U_i) = V$ and $f|_{U_i} : U_i \to V$ is a diffeomorphism. Hence $|f^{-1}(y)| \leq |f^{-1}(y')|$, for all $y' \in V$, and assuming that there exists a sequence $\{y_n\}_{n \in \mathbf{N}} \subseteq V$ such that $y_n \to y$ and $|f^{-1}(y_n)| > |f^{-1}(y)|$, it follows that one can find $x_n \in M \backslash (\bigcup_{i=1}^p U_i)$ such that $f(x_n) = y_n$. Because M is compact, $\{x_n\}_{n \in \mathbf{N}}$ has a convergent subsequence denoted in the same way, that is, there exists $x_0 = \lim\limits_{n \to \infty} x_n$. But since $x_n \in M \backslash (\bigcup_{i=1}^p U_i)$ for all $n \in \mathbf{N}$, it follows on one hand that $x_0 \in M \backslash (\bigcup_{i=1}^p U_i)$, and on the other hand, passing to the limit in the equality $f(x_n) = y_n$, that $f(x_0) = y$. This means that $x_0 \in \{x_1, x_2, \dots, x_p\} \subseteq \bigcup_{i=1}^p U_i$. Therefore there exists an open neighbourhood $V_y \subset V$ such that $|f^{-1}(y)| = |f^{-1}(y')|$ for all $y' \in V_y$.

Further on, take two arbitrary points $y,\ y' \in N \backslash B(f)$ and consider a continuous path $\gamma : [0,1] \to N \backslash B(f)$ joining y to y', i.e. $\gamma(0) = y,\ \gamma(1) = y'$ ($N \backslash B(f)$ is connected, $B(f)$ being a finite set), and cover $\gamma([0,1])$ with a finite number of above constructed open sets by means of which one can easily prove that $|f^{-1}(y)| = |f^{-1}(y')|$, and the theorem is completly proved. $\square$

2. Some homotopical aspects

For $r > 0$ and $n \in \mathbf{N}^*$ denote by D_r^n and S_r^{n-1} the open disk and the sphere respectively, having the center at the origin of the space $\mathbf{R}^n$ and radius r. D_1^n and S_1^{n-1} will be simply denoted by D^n and S^{n-1} respectively.

For $x_0 \in D^n$, define the continuous mapping $h_{x_0} : \mathbf{R}^n \backslash \{x_0\} \to \mathbf{R}^n \backslash \{x_0\}$ given by

$$h_{x_0}(x) = \begin{cases} x & \text{if } x \in \mathbf{R}^n \backslash D^n, \\ x_0 + \alpha(x)(x - x_0) & \text{if } x \in \bar{D}^n, \end{cases}$$

where

$$\alpha(x) = \left\langle \frac{x_0}{\| x - x_0 \|}, \frac{x_0 - x}{\| x - x_0 \|} \right\rangle + \sqrt{\left\langle \frac{x_0}{\| x - x_0 \|}, \frac{x_0 - x}{\| x - x_0 \|} \right\rangle^2 + \frac{1 - \| x_0 \|^2}{\| x - x_0 \|^2}}.$$

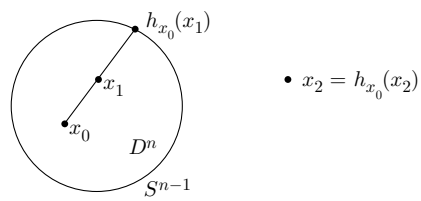

FIGURE 1.

The mapping h_{x_0} acts on $\mathbf{R}^n\backslash\{x_0\}$ as in Figure 1.

Let M be an n-dimensional manifold and $c = (U, \varphi)$ be a local chart of M such that $\bar{D}^n \subseteq \varphi(U)$. Denote by D_φ and by S_φ the sets $\varphi^{-1}(D^n)$ and $\varphi^{-1}(S^{n-1})$ respectively. For $x_0 \in D_\varphi$ define the continuous mapping $h_{c,x_0} : M\backslash\{x_0\} \to M\backslash\{x_0\}$ given by

$$h_{c,x_0}(x) = \begin{cases} x & \text{if } x \in M\backslash D_\varphi, \\ (\varphi^{-1} \circ h_{\varphi(x_0)}\Big|_{\varphi(U)} \circ \varphi)(x) & \text{if } x \in U. \end{cases}$$

Remarks. (i) $h_{x_0}(D^n\backslash\{x_0\}) = S^{n-1}$ and $h_{x_0}\Big|_{S^{n-1}} = id_{S^{n-1}}$.

(ii) $h_{x_0} \simeq_{H^{x_0}} id_{\mathbf{R}^n\backslash\{x_0\}}$ (rel $\mathbf{R}^n\backslash D^n$), where $H^{x_0} : (\mathbf{R}^n\backslash\{x_0\})\times[0,1] \to \mathbf{R}^n\backslash\{x_0\}$ is given by $H^{x_0}(x,t) = (1-t)x + th_{x_0}(x)$.

(iii) $h_{c,x_0}(D_\varphi\backslash\{x_0\}) = S_\varphi$ and $h_{c,x_0}\Big|_{S_\varphi} = id_{S_\varphi}$.

(iv) Let $c_1 = (U_1, \varphi_1), \dots, c_k = (U_k, \varphi_k)$ be local charts in M such that for $i \neq j$ $U_i \cap U_j = \emptyset$ and $D^n \subseteq \bigcap_{i=1}^k \varphi_i(U_i)$. If $x_i \in D_{\varphi_i},\ i \in \{1, 2, \dots, k\}$, then

$$g_{c_1,x_1} \circ \dots \circ g_{c_k,x_k} = g_{c_{\pi(1)},x_{\pi(1)}} \circ \dots \circ g_{c_{\pi(k)},x_{\pi(k)}},$$

where π is an element of the symmetric group of the set $\{1, 2, \dots, k\}$ and

$$g_{c_i,x_i} : M\backslash\{x_1, \dots, x_k\} \to M\backslash\{x_1, \dots, x_k\}$$

are given by $g_{c_i,x_i} = h_{c_i,x_i}\Big|_{M\backslash\{x_1,\dots,x_k\}}$, for all $i \in \{1, 2, \dots, k\}$.

(v) Under the conditions of the statement (iv), it is also true that

$$g_{c_1,x_1} \circ \dots \circ g_{c_k,x_k} \simeq_{G^{x_1\cdots x_k}_{c_1\cdots c_k}} id_{M\backslash\{x_1,\dots,x_k\}}(\text{ rel } M\backslash \bigcup_{i=1}^k D_{\varphi_i})$$

where $G^{x_1,\cdots x_k}_{c_1\cdots c_k} : (M\backslash\{x_1, \dots, x_k\}) \times [0,1] \to M\backslash\{x_1, \dots, x_k\}$ is given by

$$G^{x_1\cdots x_k}_{c_1\cdots c_k}(x,t) = \begin{cases} x & \text{if } x \in M\backslash \bigcup_{i=1}^k D_{\varphi_i}, \\ \varphi_1^{-1}(H^{\varphi_1(x_1)}(\varphi_1(x), t)) & \text{if } x \in \bar{D}_{\varphi_1}, \\ \vdots & \\ \varphi_k^{-1}(H^{\varphi_k(x_k)}(\varphi_k(x), t)) & \text{if } x \in \bar{D}_{\varphi_k}. \end{cases}$$

Lemma 2.1. *Let M_1 be a differentiable manifold with boundary, M_2 be a differentiable manifold without boundary, C be a closed subset of M_2 and $f : M_1 \to M_2$ be a continuous mapping such that $f(\partial M_1) \subseteq M_2\backslash C$. Then there exists a homotopy $H : M_1 \times [0,1] \to M_2$ such that $f = H(\cdot, 0)$, $H_t(\partial M_1) \subseteq M_2\backslash C$ for all $t \in [0,1]$ and $H(\cdot, 1)$ is a differentiable mapping.*

Proof. There exists a homotopy $K : \partial M_1 \times [0,1] \to M_2 \backslash C$ such that $K(\cdot,0) = f\big|_{\partial M_1}$ and $K(\cdot,1)$ is a differentiable mapping (see [Go, Proposition 4.6, pp. 64]). Consider the homotopy $H^2 : M_1 \times [0,1] \to M_2$:

$$H^2(x,t) = \begin{cases} K((\pi_1 \circ Q^{-1})(x), 1) & \text{if } x \in Q(\partial M_1 \times [0, 1-t]), \\ K\big((\pi_1 \circ Q^{-1})(x), \frac{2-(\pi_2 \circ Q^{-1})(x)}{t+1}\big) & \text{if } x \in Q(\partial M_1 \times [1-t, 2]), \\ f\big(Q((\pi_1 \circ Q^{-1})(x), 3(\pi_2 \circ Q^{-1})(x) - 6)\big) & \text{if } x \in Q(\partial M_1 \times [2,3]), \\ f(x), & \text{if } x \in M_1 \backslash Q(\partial M_1 \times [0,3)), \end{cases}$$

where $Q : \partial M_1 \times [0,\infty) \to U \subseteq M_1$ is a collar neighbourhood of the boundary ∂M_1 (U is an open neighbourhood of the boundary ∂M_1 and Q is a diffeomorphism such that $Q(x,0) = x,\ \forall\, x \in \partial M_1$) and $\pi_1 : \partial M_1 \times [0,\infty) \to \partial M_1,\ \ \pi_2 : \partial M_1 \times [0,\infty) \to [0,\infty)$ are the projections.

Consider the homotopy $H^3 : M_1 \times [0,1] \to M_2$:

$$H^3(x,t) = \begin{cases} K\big((\pi_1 \circ Q^{-1})(x), 1 - t - \frac{1}{2}(\pi_2 \circ Q^{-1})(x)\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \in Q(\partial M_1 \times [0, 2(1-t)]), \\ f\big(Q((\pi_1 \circ Q^{-1})(x), \frac{3(\pi_2 \circ Q^{-1})(x) - 6(1-t)}{1+2t})\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \in Q(\partial M_1 \times [2(1-t), 3]), \\ f(x), \qquad\qquad\qquad\qquad \text{if } x \in M_1 \backslash Q(\partial M_1 \times [0,3)), \end{cases}$$

and observe that $H^3(\cdot,0) = H^2(\cdot,1)$ and $H^3(\cdot,1) = f$. Because $K(\cdot,1)$, π_1, $Q^{-1}\Big|_{Q(\partial M_1 \times [0,1))}$ are differentiable mappings, it follows that

$$H^2(\cdot,0)\Big|_{Q(\partial M_1 \times [0,1))} = K(\cdot,1) \circ \pi_1 \circ \Big(Q^{-1}\Big|_{Q(\partial M_1 \times [0,1))}\Big)$$

is differentiable on the open neighbourhood $Q(\partial M_1 \times [0,1))$ of ∂M_1. Hence, there exists a homotopy $H^1 : M_1 \times [0,1] \to M_2$ such that $H^1(\cdot,0)$ is a differentiable mapping, $H^1(\cdot,0) \simeq_{H^1} H^1(\cdot,1)(\text{rel } \partial M_1)$ and $H^1(\cdot,1) = H^2(\cdot,0)$ (see [Go, Théorème 4.5, pp. 64]). Choose $H : M_1 \times [0,1] \to M_2$ in the following way:

$$H(x,t) = \begin{cases} H^3(x, 1-3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ H^2(x, 2-3t) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ H^1(x, 3-3t) & \text{if } \frac{2}{3} \le t \le 1. \end{cases} \quad \square$$

Theorem 2.2. *Let M be an n-dimensional differentiable manifold ($\partial M = \emptyset$) and A be a discrete subset of M whose derived set A' is also discrete. If P is a compact differentiable k-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $f : P \to M$ is a continuous mapping such that $f(\partial P) \subseteq M \backslash \bar{A}$, then there exists a continuous mapping $g : P \to M$ such that $g(P) \subseteq M \backslash \bar{A}$, $g|_{\partial P} = f|_{\partial P}$ and $f \simeq g(\text{ rel } \partial P)$.*

Proof. According to Lemma 2.1 there exists a homotopy $H : P \times [0,1] \to M$ such that $f = H(\cdot,0)$, $g_1 = H(\cdot,1)$ is a differentiable mapping and $H_t(\partial P) \subseteq M \backslash \bar{A}$ for all $t \in [0,1]$.

Because A' is a closed and discrete subset of M and $g_1(P)$ is compact, it follows that $g_1(P) \cap A'$ is a finite set. Let us assume that $g_1(P) \cap A' = \{x_1, x_2, \ldots, x_l\}$. It

is easily seen that there exist the local charts $c_1 = (U_1, \varphi_1), \dots, c_l = (U_l, \varphi_l)$ such that:

(i) $x_i \in U_i$ and $\varphi_i(x_i) = 0$ for all $i \in \{1, 2, \dots, l\}$;

(ii) $U_i \cap U_j = \emptyset$ for $i \neq j$;

(iii) $H_t(\partial P) \cap U_i = \emptyset$ for all $t \in [0, 1]$ and all $i \in \{1, \dots, l\}$.

Because A and A' are discrete sets and M has a countable basis, it follows that both of them are countable, which means that the closure $\bar{A} = A \cup A'$ is countable too. We can therefore say that $\varphi_i(U_i \cap \bar{A})$ is countable for each $i \in \{1, 2, \dots, l\}$. But this means that there exists $r > 0$ such that $\bar{D}^n_r \subseteq \bigcap_{i=1}^l \varphi_i(U_i)$ and $S^{n-1}_r \cap \varphi_i(U_i \cap \bar{A}) = \emptyset$ for all $i \in \{1, 2, \dots, l\}$. Making, if need be, easy modifications of the mappings φ_i, we can assume that $r = 1$. Therefore, $S_{\varphi_i} \cap \bar{A} = \emptyset$ or, equivalently, $S_{\varphi_i} \subseteq M \backslash \bar{A}$. Therefore the following inclusions

(iv) $\bar{D}^n \subseteq \bigcap_{i=1}^l \varphi_i(U_i)$ and $S_{\varphi_i} \subseteq M \backslash \bar{A}$

hold. Obviously $[g_1(P) \backslash (\bigcup_{i=1}^l U_i)] \cap \bar{A}$ is a finite set; for instance $[g_1(P) \backslash (\bigcup_{i=1}^l U_i)] \cap \bar{A} = \{x_{l+1}, \dots, x_{l+l'}\}$. Using the same kind of arguments as above, there exist the local charts $c_{l+1} = (U_{l+1}, \varphi_{l+1}), \dots, c_{l+l'} = (U_{l+l'}, \varphi_{l+l'})$ satisfying the following conditions:

(i′) $x_{l+i} \in U_{l+i}$ and $\varphi_{l+i}(x_{l+i}) = 0$ for all $i \in \{1, \dots, l'\}$;

(ii′) $U_{l+i} \cap U_{l+j} = \emptyset$ and $U_{l+i} \cap U_k = \emptyset$ for all $i, j \in \{1, \dots, l'\}$, $k \in \{1, \dots, l\}$ and $i \neq j$, $i \neq k$;

(iii′) $H_t(\partial P) \cap U_{l+i} = \emptyset$ for all $t \in [0, 1]$ and all $i \in \{1, \dots, l'\}$;

(iv′) $\bar{D}^n \subseteq \bigcap_{i=1}^{l'} \varphi_{l+i}(U_{l+i})$ and $S_{\varphi_{l+i}} \subseteq M \backslash \bar{A}$ for all $i \in n\{1, \dots, l'\}$.

Because g_1 is a differentiable mapping, we obtain, using a weak version of Sard's theorem for the mapping $g_1|_{int\, P}$, that $D_{\varphi_i} \backslash g_1(P)$ is nonempty for every $i \in \{1, 2, \dots, l + l'\}$. For $i \in \{1, 2, \dots, l + l'\}$, let us consider $y_i \in D_{\varphi_i} \backslash g_1(P)$ and the mappings

$$g_{c_i, y_i} : M \backslash \{y_1, \dots, y_{l+l'}\} \to M \backslash \{y_1, \dots, y_{l+l'}\},$$

$$h : P \to M,\ h(x) = (j \circ g_{c_1, y_1} \circ \cdots \circ g_{c_{l+l'}, y_{l+l'}})(g_1(x))$$

where $j : M \backslash \{y_1, \dots, y_{l+l'}\} \to M$ is the inclusion map. Obviously, $h(P) \subseteq M \backslash \bar{A}$, $h|_{\partial P} = g_1|_{\partial P}$ and $h \simeq_{G'} g_1(\text{ rel } \partial P)$, where $G' : P \times [0, 1] \to M$ is given by $G'(x, t) = G^{y_1 \cdots y_{l+l'}}_{c_1 \cdots c_{l+l'}}(g_1(x), t)$.

By the transitivity of the relation " $\simeq$ ", one can conclude that $f \simeq_{H'} h$ where $H' : P \times [0, 1] \to M$ is given by

$$H'(x, t) = \begin{cases} H(x, 2t), & 0 \leq t \leq \frac{1}{2}, \\ G'(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Consider the following two homotopies $\psi : P \times [0, 1] \to P$, and $G : P \times [0, 1] \to M$ given by:

$$\psi(x, t) = \begin{cases} x & \text{if } x \in P \backslash Q(\partial P \times [0, 2)), \\ Q((\pi_1 \circ Q^{-1})(x), \frac{2}{2-t}(\pi_2 \circ Q^{-1})(x) + \frac{2t}{t-2}) & \text{if } x \in Q(\partial P \times [t, 2]), \\ (\pi_1 \circ Q^{-1})(x) & \text{if } x \in Q(\partial P \times [0, t]), \end{cases}$$

$$G(x, t) = \begin{cases} H'(\psi(x, t), t) & \text{if } x \in P \backslash Q(\partial P \times [0, t)), \\ H'(Q^{-1}(x)) & \text{if } x \in Q(\partial P \times [0, t]), \end{cases}$$

where $Q : \partial P \times [0, \infty) \to U \subset P$ is a collar neighbourhood of ∂P and $\pi_1 : \partial P \times [0, \infty) \to \partial P$, $\pi_2 : \partial P \times [0, \infty) \to [0, \infty)$ are obviously the projections.

Denoting $G(\cdot,1)$ by g and observing that $G(\cdot,0)=f$ it can be easily seen that $f \simeq_G g(\text{ rel } \partial P)$ and also that $g(P) \subseteq M\backslash\bar{A}$, and the theorem is completely proved. □

Proposition 2.3. *Let M be an n-dimensional differentiable manifold ($n \geq 2$, $\partial M = \emptyset$) and A be a discrete subset of M whose derived set A' is also discrete. If M is connected, then $M\backslash\bar{A}$ is also connected, and the inclusion $i : M\backslash\bar{A} \to M$ is $(n-1)$-connected; that is, the homomorphism $i_q : \pi_q(M\backslash\bar{A}) \to \pi_q(M)$, induced by the inclusion, is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$.*

Proof. The connectedness of $M\backslash\bar{A}$ follows easily from Theorem 2.2, taking the particular case $P=[0,1]$. Using the exact homotopy sequence

$$\cdots \to \pi_{i+1}(M, M\backslash\bar{A}) \to \pi_i(M\backslash\bar{A}) \xrightarrow{i_*} \pi_i(M) \to \pi_i(M, M\backslash\bar{A}) \to \cdots$$

it is enough to show that $\pi_r(M, M\backslash\bar{A}) = 0$ for all $r \in \{1,2,\dots,n-1\}$. But this is an immediate consequence of Theorem 2.2 and of the fact that $[\alpha] \in \pi_r(M, M\backslash\bar{A})$ is zero if and only if there exists $\beta \in [\alpha]$ such that $\beta(D^r) \subseteq M\backslash\bar{A}$. □

3. Basic result

Using the results previously obtained, in this section we shall prove the already announced sufficient conditions, in terms of homotopy groups, in order that $\varphi(M,N)$ be infinite, for the given pair (M,N) of differentiable manifolds.

Theorem 3.1. *Let M, N be compact connected differentiable manifolds having the same dimension m. In these conditions the following statements are true:*

(i) *If $m \geq 3$ and $\pi_1(N)$ has no subgroup isomorphic with $\pi_1(M)$, then $\varphi(M,N) = \infty$;*

(ii) *If $m \geq 4$ and $\pi_q(M) \ncong \pi_q(N)$ for some $q \in \{2,3,\dots,m-2\}$, then $\varphi(M,N) = \infty$.*

Proof. We have to show that any differentiable mapping $f : M \to N$ has infinitely many critical points. If f is not surjective, this follows easily by Theorem 1.1. Assume that $f : M \to N$ is surjective and that f has finitely many critical points. Hence, by Theorem 1.2,

$$g = f|_{M\backslash f^{-1}(B(f))} : M\backslash f^{-1}(B(f)) \to N\backslash B(f)$$

is a covering map, which means that

$$g_1 : \pi_1(M\backslash f^{-1}(B(f))) \to \pi_1(N\backslash B(f))$$

is a monomorphism and

$$g_q : \pi_q(M\backslash f^{-1}(B(f))) \to \pi_q(N\backslash B(f))$$

are isomorphisms for all $q \geq 2$. On the other hand, by Theorem 1.2 and Proposition 2.3 it follows that the homomorphisms $i_q : \pi_q(M\backslash f^{-1}(B(f))) \to \pi_q(M)$ and $j_q : \pi_q(N\backslash B(f)) \to \pi_q(N)$ induced by the inclusions $i : M\backslash f^{-1}(B(f)) \to M$ and $j : N\backslash B(f) \to N$ are isomorphisms for all $q \in \{0,1,\dots,m-2\}$. From the commutative

diagram

$$\begin{array}{ccc} M\backslash f^{-1}(B(f)) & \xrightarrow{g} & N\backslash B(f) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ M & \xrightarrow{f} & N \end{array} \qquad j\circ g = f\circ i$$

we get the following commutative diagrams:

$$\begin{array}{ccc} \pi_q(M\backslash f^{-1}(B(f))) & \xrightarrow{g_q} & \pi_q(N\backslash B(f)) \\ {\scriptstyle i_q}\downarrow & & \downarrow{\scriptstyle j_q} \\ \pi_q(M) & \xrightarrow{f_q} & \pi_q(N) \end{array} \qquad j_q\circ g_q = f_q\circ i_q.$$

(i) In the case when $q=1$, because $f_1\circ i_1 = j_1\circ g_1$, $i_1,\ j_1$ are isomorphisms and g_1 is a monomorphism, it follows that $f_1 = j_1\circ g_1\circ i_1^{-1}$ is a monomorphism, which is a contradiction with the hypothesis of the statement (i).

(ii) For $q\in\{2,\dots,m-2\}$, $i_q,\ j_q,\ g_q$ are isomorphisms which together with $f_q\circ i_q = j_q\circ g_q$ leads to the conclusion that $f_q = j_q\circ g_q\circ i_q^{-1}$ are isomorphisms for all $q\in\{2,\dots,m-2\}$, which is a contradiction with the hypothesis of the statement (ii). □

Corollary 3.2. (i) *If M and N are compact diferentiable manifolds having the same dimension $m\geq 3$ and $|\pi_1(M)|>|\pi_1(N)|$, then $\varphi(M,N)=\infty$. So, in particular $\varphi(T^m,S^m)=\infty$ and $\varphi(P^n(\mathbf{R})\times P^k(\mathbf{R}),P^{n+k}(\mathbf{R}))=\infty$ where n, k are two natural numbers such that $n+k\geq 3$.*

(ii) *Under the same conditions on the manifolds M,N as in the statement above and if $\pi_1(M)$ and $\pi_1(N)$ are finite groups such that $(|\pi_1(M)|,|\pi_1(N)|)=1$, one gets that $\varphi(M,N)=\infty$.*

(iii) *If m, k are two natural numbers such that $n,\ k\geq 2$, then*

$$\varphi(T^{n+k},T^n\times S^k)=\infty,\ \varphi(T^n\times S^k,T^{n+k})=\infty,$$

$$\varphi(S^{n+k},S^n\times S^k)=\infty,\ \varphi(S^n\times S^k,S^{n+k})=\infty,$$

where T^p denotes the p dimensional torus $\underbrace{S^1\times\cdots\times S^1}_{p\ times}$.

4. The equivariant case

In this section we shall prove an equivariant version of Theorem 3.1.

Let G be a Lie group, M be a manifold and $\varphi: G\times M\to M,\ (g,x)\mapsto gx$ be a smooth action of G on M. The triple (G,M,φ) is called a G-manifold. If the action of G on M is free, recall that M/G can be endowed with a differential structure such that the canonical projection $\pi_M: M\to M/G$ is a smooth G bundle. A mapping $f: M\to N$ between the G-manifolds M and N is said to be G-equivariant if $f(gx)=gf(x)$ for all $g\in G$ and all $x\in M$. If M and N are two G-manifolds and $f: M\to N$ is smooth G-equivariant, denote by $\bar f: M/G\to N/G$ the smooth mapping which makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M/G & \xrightarrow{\bar f} & N/G \end{array} \qquad \pi_N\circ f = \bar f\circ\pi_M.$$

Theorem 4.1. *Let G be a connected Lie group and M, N be two compact connected G-manifolds such that $\dim M = \dim N$ and the actions of G on M and N are free. Let k be the common dimension of the manifolds M/G and N/G. Under these conditions the following assertions are true:*

(i) *If $k \geq 3$, $\pi_1(M)$, $\pi_1(N)$ are commutative groups and $\pi_1(N)$ has no subgroup isomorphic with $\pi_1(M)$, then any G-equivariant mapping $f : M \to N$ has infinitely many critical orbits.*

(ii) *If $k \geq 4$ and $\pi_q(M) \ncong \pi_q(N)$ for some $q \in \{2, \dots, k-2\}$, then any G-equivariant mapping $f : M \to N$ has infinitely many critical orbits.*

Proof. For both of the above assertions it is enough to show that the associated mapping $\bar{f} : M/G \to N/G$ of an arbitrary G-equivariant mapping $f : M \to N$ has infinitely many critical points. For this purpose, we can assume that $\bar{f} : M/G \to N/G$ is surjective because otherwise the fact that $\bar{f}$ has infinitely many critical points follows easily from Theorem 1.1. Supposing that $\bar{f} : M/G \to N/G$ has finitely many critical points, it follows, by Theorem 1.2, that

$$\bar{g} = \bar{f}\big|_{M/G \setminus \bar{f}^{-1}(B(\bar{f}))} : M/G \Big\backslash \bar{f}^{-1}(B(\bar{f})) \to N/G \Big\backslash B(\bar{f})$$

is a covering map, which means that

$$\bar{g}_q : \pi_q(M/G \Big\backslash \bar{f}^{-1}(B(\bar{f}))) \to \pi_q(N/G \Big\backslash B(\bar{f}))$$

is a monomorphism if $q = 1$ and it is an isomorphism if $q \geq 2$. The commutative diagrame

$$\begin{array}{ccc} M/G \Big\backslash \bar{f}^{-1}(B(\bar{f})) & \xrightarrow{\bar{g}} & N/G \Big\backslash B(\bar{f}) \\ \bar{i}\Big\downarrow & & \Big\downarrow \bar{j} \\ M/G & \xrightarrow{\bar{f}} & N/G \end{array} \qquad \bar{j} \circ \bar{g} = \bar{f} \circ \bar{i}$$

supplies the following commutative diagrams:

$$\begin{array}{ccc} \pi_q(M/G \Big\backslash \bar{f}^{-1}(B(\bar{f}))) & \xrightarrow{\bar{g}_q} & \pi_q(N/G \Big\backslash B(\bar{f})) \\ \bar{i}_q\Big\downarrow & & \Big\downarrow \bar{j}_q \\ \pi_q(M/G) & \xrightarrow{\bar{f}_q} & \pi_q(N/G) \end{array} \qquad \bar{j}_q \circ \bar{g}_q = \bar{f}_q \circ \bar{i}_q.$$

By Proposition 2.3, it follows that for $q \in \{1, \dots, k-2\}$ the homomorphisms $\bar{i}_q$ and $\bar{j}_q$ induced by inclusions $\bar{i}$ and $\bar{j}$ are isomorphisms, and they are epimorphisms for $q = k-1$. Therefore $\bar{f}_q$ is a monomorphism for $q = 1$, it is an isomorphism if $q \in \{2, \dots, k-2\}$ and $\bar{f}_{k-1}$ is an epimorphism. Consider the following ladder with commutative rectangles:

$$\begin{array}{ccccccccccc} \to & \pi_q(M/G) & \longrightarrow & \pi_{q-1}(G) & \longrightarrow & \pi_{q-1}(M) & \longrightarrow & \pi_{q-1}(M/G) & \longrightarrow & \pi_{q-2}(G) & \to \\ & \Big\downarrow \bar{f}_q & & \Big\downarrow & & \Big\downarrow f_{q-1} & & \Big\downarrow \bar{f}_{q-1} & & \Big\downarrow & \\ \to & \pi_q(N/G) & \longrightarrow & \pi_{q-1}(G) & \longrightarrow & \pi_{q-1}(N) & \longrightarrow & \pi_{q-1}(N/G) & \longrightarrow & \pi_{q-2}(G) & \to \end{array}$$

where the rows are the exact homotopy sequences of the principal fibrations $G \hookrightarrow M \to M/G$, $G \hookrightarrow N \to N/G$. Using successively the five lemma for $q = 2$ and then for $q \in \{3, \dots, k-1\}$ in the above diagram, it follows that f_1 is a monomorphism and f_q are isomorphisms for all $q \in \{2, \dots, k-2\}$, which contradicts the hypothesis of (i) and that of (ii) respectively, and the theorem is completely proved. $\square$

As an application of Theorem 4.1, consider the following free actions of S^1 on S^{2n+1} and T^{2n+1} respectively:

$$S^1 \times T^{2n+1} \to T^{2n+1},\ (z,(z_1,\dots,z_{2n+1})) \mapsto (z^{a_1}z_1,\dots,z^{a_{2n+1}}z_{2n+1}),$$

$$S^1 \times S^{2n+1} \to S^{2n+1},\ (z,(z_1,\dots,z_n)) \mapsto (zz_1.\dots,zz_n),$$

where $n, a_1, \dots, a_{2n+1}$ are natural numbers such that $n \geq 1$, and $(a_1,\dots,a_{2n+1}) = 1$.

Theorem 4.1 (i) implies that any S^1-equivariant mapping $f : T^{2n+1} \to S^{2n+1}$ has an infinite number of critical orbits. The fact that any S^1-equivariant mapping $g : S^{2n+1} \to T^{2n+1}$ has infinitely many critical orbits is proved in the paper [Pi2].

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