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SETS OF MINIMAL HAUSDORFF DIMENSION FOR QUASICONFORMAL MAPS

JEREMY T. TYSON

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ABSTRACT. For any $1 \le \alpha \le n$, there is a compact set $E \subset \mathbb{R}^n$ of (Hausdorff) dimension α whose dimension cannot be lowered by any quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$. We conjecture that no such set exists in the case $\alpha < 1$. More generally, we identify a broad class of metric spaces whose Hausdorff dimension is minimal among quasisymmetric images.

1. Introduction

Quasiconformal homeomorphisms of a Euclidean space $\mathbb{R}^n$, $n \ge 2$, can distort the Hausdorff dimensions of subsets. While the dimensions of sets of Hausdorff dimension zero or n must be preserved, Gehring and Väisälä [5, pp. 505-506] construct, for any $\beta \in (0, n)$, a compact set $E_\beta \subset \mathbb{R}^n$ with $\dim E_\beta = \beta$ and, for any $\beta, \beta' \in (0, n)$, a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $E_{\beta'} = fE_\beta$. Recently, Bishop [2] showed that the dimension of any compact set $E \subset \mathbb{R}^n$ of positive dimension can be raised arbitrarily close to n by a quasiconformal (quasisymmetric if $n = 1$) homeomorphism of $\mathbb{R}^n$.

On the other hand, for certain values of $\alpha \in (0, n)$, one can construct compact sets $E \subset \mathbb{R}^n$ of dimension α whose dimension cannot be lowered by any quasiconformal mapping of the ambient space. For example, consider the closed unit k-cube $[0, 1]^k$, $k = 1, 2, \dots , n-1$. These examples have Hausdorff dimension equal to their topological dimension and thus the Hausdorff dimension cannot be lowered by **any** homeomorphism of $\mathbb{R}^n$. A different example (arising from the theory of removable sets for quasiconformal maps) was given by Bishop [1] in the case $n = 3$, $\alpha = 2$. Bishop [2] asks whether there exist examples of this phenomenon for all $0 < \alpha < n$. The following result is a simple consequence of the main theorem in this paper:

Theorem 1.1. *Let $1 \le \alpha \le n$. There is a compact set $E \subset \mathbb{R}^n$ with Hausdorff dimension α so that $\dim fE \ge \alpha$ for all quasiconformal maps $f : \mathbb{R}^n \to \mathbb{R}^n$.*

When $\alpha > 1$ we may take the set E to be the Cartesian product of any Ahlfors $(\alpha - 1)$-regular set in $\mathbb{R}^{n-1}$ with the unit interval $[0, 1]$; for the definition of Ahlfors

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regularity, see section 2 (for the existence of such sets, note that the set E_β in the above example of Gehring and Väisälä is β-regular).

We will prove a more general result (Theorem 3.4) using the machinery of the generalized (or discrete) modulus introduced in [9] and developed in [13]. The restriction to subsets of a Euclidean space is not necessary; we give a condition on metric spaces which is sufficient to guarantee that the Hausdorff dimension is minimal among quasisymmetric images (recall that in $\mathbb{R}^n$, $n \geq 2$, the classes of quasiconformal and quasisymmetric maps agree).

It should be emphasized that Theorem 1.1 is not new; the result appears in works of Pansu [9] and Bourdon [3] (see, e.g., [9, Example 3.4] and [3, Example 1.7(b)]). These authors give a family of conditions (Lemma 3.9) on an arbitrary metric space which imply nontrivial lower bounds on the Hausdorff dimensions of quasisymmetric images. With an additional (weak) assumption, we show in Proposition 3.11 that the hypotheses of Theorem 3.4 are implied by the "best" condition of Bourdon and Pansu. The above-mentioned examples which illustrate Theorem 1.1 satisfy this "best" condition of Bourdon-Pansu.

The restriction to dimensions larger than one is necessary in our proof, which relies on the geometric characterization of quasiconformality in terms of curve families. We conjecture that no such example exists for dimensions strictly smaller than one, more precisely:

Conjecture 1.2. *Let $n \geq 1$ and $\epsilon > 0$. If a compact set $E \subset \mathbb{R}^n$ has Hausdorff dimension strictly smaller than one, then there exists a quasiconformal (quasisymmetric if $n = 1$) map $f : \mathbb{R}^n \to \mathbb{R}^n$ so that fE has Hausdorff dimension smaller than ϵ.*

Note that such a set must be totally disconnected. For related results, see [12].

2. Notation and definitions

In a metric space X, we denote by $\overline{B}(x,r)$ the closed ball with center $x \in X$ and radius $r > 0$. For $Y \subset X$, we write $\operatorname{diam} Y$ for the diameter of Y. If $B = \overline{B}(x,r)$ is a ball in X, we write CB for the *dilated ball* $\overline{B}(x,Cr)$.

For $x \in X$ and $r > 0$, we have $\operatorname{diam} \overline{B}(x,r) \leq 2r$, but no lower bound for $\operatorname{diam} \overline{B}(x,r)$ need hold in general. We call X (c_1-)*uniformly perfect*, $0 < c_1 \leq 2$, if $\operatorname{diam} \overline{B}(x,r) \geq c_1 r$ for every $x \in X$ and $0 < r \leq \operatorname{diam} X$. Every connected space is 1-uniformly perfect.

For $\alpha > 0$, we denote by $\mathcal{H}_\alpha$ the Hausdorff α-measure on X and by $\dim X$ the Hausdorff dimension of X [4, §2.10.2(1)]. For connected sets E, we have the relation $\operatorname{diam} E \leq \mathcal{H}_1(E)$ [4, §2.10.12].

For $\alpha > 0$, we call a Borel measure μ on X *(Ahlfors) α-regular* if there exists a constant $C_0 \geq 1$ (called a *regularity constant for X*) so that

$$\frac{1}{C_0} r^\alpha \leq \mu(\overline{B}(x,r)) \leq C_0 r^\alpha \tag{2.1}$$

for all $x \in X$ and $0 < r < \operatorname{diam} X$. If μ is a Borel measure on X satisfying (2.1), then $\mathcal{H}_\alpha$ also satisfies (2.1) (possibly with a different regularity constant) and μ and $\mathcal{H}_\alpha$ are equivalent. Thus α-regular spaces have Hausdorff dimension α. Every α-regular space with regularity constant C_0 is c_1-uniformly perfect for each $c_1 < C_0^{-2/\alpha}$.

We call a homeomorphism $f : X \to Y$ between metric spaces (η-)*quasisymmetric*, where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism, if

$$|x - y| \leq t\,|x - z| \Rightarrow |f(x) - f(y)| \leq \eta(t)\,|f(x) - f(z)| \tag{2.2}$$

for any $x, y, z \in X$ and $t > 0$ [11]. By making η larger if necessary, we can (and will) always assume that $\eta(t) \geq t$ for $t > 0$ and that both f and f^{-1} satisfy (2.2). If X is c_1-uniformly perfect and $f : X \to Y$ is η-quasisymmetric, then Y is c_1'-uniformly perfect for some $c_1' = c_1'(c_1, \eta) > 0$.

The Hausdorff dimension of a metric space is not a quasisymmetric invariant. Following Pansu [9, Définition 3.1], we define the *conformal dimension* of a metric space X to be

$$\mathcal{C} \dim X = \inf_Y \dim Y,$$

where the infimum is taken over all spaces Y which are quasisymmetrically equivalent to X. For any space X, $\mathcal{C} \dim X \geq \dim_T X$, where $\dim_T X$ denotes the *topological dimension* of X [7].

3. The main result

Let (X, μ) be a metric measure space. For a family Γ of curves in X and $p \geq 1$, we denote by $\operatorname{Mod}_p \Gamma$ the *p-modulus* of Γ [14, §6.1], [6, §2.3]. (Throughout this paper, we restrict ourselves to consider only curves whose parametrization is one-to-one.)

Definition 3.1. We say that X has *nontrivial p-modulus*, $p \geq 1$, if $\operatorname{Mod}_p \Gamma > 0$ for some family of curves Γ in X.

Example 3.2. Let $\alpha > 1$. Let Z be any compact $(\alpha - 1)$-regular metric space and let X be the product space $Z \times [0, 1]$ (endowed with any of its standard bi-Lipschitz equivalent metrics). Then X is a compact, α-regular space and the family of curves

$$\Gamma = \{\gamma_z : [0, 1] \to X | \gamma_z(t) = (z, t), z \in Z\}$$

has positive α-modulus. Note that for any $n \geq 2$ and $1 < \alpha \leq n$, we may choose $Z \subset \mathbb{R}^{n-1}$ (see the introduction) and so $X \subset \mathbb{R}^n$.

When $1 < \alpha < 2$, the preceding example is not connected. We next give connected examples for each $1 < \alpha \leq n$.

Example 3.3. Let $1 < \alpha \leq n$ and choose a compact $(\alpha - 1)$-regular Borel set $Z \subset S^{n-1}$. Let ν be an $(\alpha - 1)$-regular measure on Z and let

$$X = \{tz : z \in Z, 0 \leq t \leq 1\} \subset \mathbb{R}^n$$

be the *cone on* Z. Here, for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and $t \geq 0$, we write $tz = (tz_1, \ldots, tz_n)$. We define a measure μ on X as follows: for Borel sets $A \subset [0, 1]$ and $B \subset Z$, define the *spherical product set* $A \cdot B = \{tz : t \in A, z \in B\}$ and set

$$\mu(A \cdot B) = \nu(B) \cdot \int_A r^{n-1} dr.$$

The collection of spherical product sets generates the Borel σ-algebra on X and μ is an α-regular Borel measure. Moreover, the family of curves

$$\Gamma = \{\gamma_z : [1/2, 1] \to X | \gamma_z(t) = tz, z \in Z\}$$

has positive α-modulus, by an argument similar to [14, Example 7.5].

Theorem 1.1 is a consequence of these examples and the following result.

Theorem 3.4. *If X is a compact α-regular space with nontrivial α-modulus, $\alpha > 1$, then $\mathcal{C}\dim X = \alpha$.*

To prove Theorem 3.4, we recall the theory of the generalized modulus, first introduced by Pansu in [9], [10] and developed by the author in [13].

Definition 3.5. Let X be a metric space and let $p > 0$. For $k \geq 1$, let $\mathcal{B}_k(X)$ denote the collection of all closed sets $A \subset X$ for which

$$\overline{B}(x,r) \subset A \subset \overline{B}(x,kr)$$

for some $x \in X$ and $r > 0$. We write $\mathcal{B}(X) = \bigcup_{k\geq 1} \mathcal{B}_k(X)$. For a set function $\varphi : \mathcal{B}(X) \to [0,\infty]$, we denote by $\Phi_{p,k}$ the Borel measure constructed by the Carathéodory construction (Method II) using the set function φ^p and the family of sets $\mathcal{B}_k(X)$. See, for example, [8, §4.1] or [4, §2.10.1].

For $l \geq 1$, we define a set function $\tilde{\varphi}_l : \mathcal{B}_l(X) \to [0,\infty]$ as follows: let $\tilde{\varphi}_l(A)$ be the supremum of the values $\varphi(\tilde{A})$ over all sets $\tilde{A}$ for which

$$\overline{B}(x,r) \subset A \subset \tilde{A} \subset \overline{B}(x,lr)$$

for some $x \in X$ and $r > 0$. Finally, we denote by $\tilde{\Phi}_{p,k,l}$ the corresponding measure constructed using the set function $\tilde{\varphi}_l^p$.

Example 3.6. The most well-known example of a measure generated by the Carathéodory construction is the Hausdorff measure $\mathcal{H}_p$, which is obtained by using the set function φ^p, where $\varphi(A) = \operatorname{diam} A$, on the family of all subsets $A \subset X$. It is easy to verify that in this case

$$\mathcal{H}_p(E) \leq \Phi_{p,k}(E) \leq 2^p \mathcal{H}_p(E) \tag{3.1}$$

for all Borel sets $E \subset X$ and all $k \geq 1$. Moreover, if X is c_1-uniformly perfect, then for this φ we have $\tilde{\varphi}_l(A) \leq (2l/c_1)\varphi(A)$ for all $A \in \mathcal{B}_l(X)$, $l \geq 1$, and so

$$\Phi_{p,k}(E) \leq \tilde{\Phi}_{p,k,l}(E) \leq (2l/c_1)^p \Phi_{p,k}(E) \tag{3.2}$$

for all Borel sets $E \subset X$ and $l \geq k \geq 1$.

Definition 3.7. Let X be a metric space and let $p > 0$. Let Γ be a family of subsets of X. For $m \geq 1, l \geq k \geq 1$, we define the *generalized p-modulus of* Γ *(with data k, l, m)* to be

$$\operatorname{mod}_{p,k,l,m} \Gamma = \inf \tilde{\Phi}_{p,k,l}(X),$$

where the infimum is taken over all set functions $\varphi : \mathcal{B}(X) \to [0,\infty]$ satisfying $\Phi_{1,m}(\gamma) \geq 1$ for all $\gamma \in \Gamma$.

This concept was introduced in [9] and further developed in [13]. We recall some basic properties of the generalized modulus. For proofs of these results, see Propositions 3.24, 4.5 and 4.7 in [13]; note that the assumption of connectedness in Propositions 4.5 and 4.7 of [13] can be weakened to uniform perfectness.

Proposition 3.8. (i) *Let $f : X \to Y$ be η-quasisymmetric and let $p > 0$. For $k, k', l, l', m, m' \geq 1$ satisfying $l' \geq \eta(l)$, $l \geq k' \geq \eta(k)$ and $m' \geq \eta(m)$ and any collection Γ of subsets of X,*

$$\operatorname{mod}_{p,k',l,m} f\Gamma \leq \operatorname{mod}_{p,k,l',m'} \Gamma.$$

(ii) *Let X be a locally compact α-regular space, $\alpha > 1$, with regularity constant C_0. For any $k, l, m \geq 1$ with $l \geq 10k$, there is a constant $C = C(C_0, \alpha, k, l)$ so that*

$$\frac{1}{C} \operatorname{Mod}_\alpha \Gamma \leq \operatorname{mod}_{\alpha,k,l,m} \Gamma \leq C \operatorname{Mod}_\alpha \Gamma$$

for any curve family Γ in X.

Before proving Theorem 3.4, we pause to discuss some earlier conditions which suffice to give lower bounds for conformal dimensions. Lemma 3.9 is not used in the proof of Theorem 3.4; we include it here in order to indicate the relationship between condition (3.3) and the hypothesis of nontrivial α-modulus in the α-regular setting. See Proposition 3.11.

The following result is Lemme 1.6 of [3]. Note that the term "quasiconformal" in [3] corresponds to our term quasisymmetric and that for Ahlfors α-regular spaces, both the packing dimension $\dim_P X$ and the Hausdorff dimension $\dim_H X$ discussed in [3, §1.5] are equal to α.

Lemma 3.9. *Let X be a compact α-regular metric space and let $\beta \geq 0$. Suppose that there exists a curve family Γ in X with the following properties:*

- *there exists $\delta > 0$ so that $\operatorname{diam} \gamma \geq \delta$ for each $\gamma \in \Gamma$,*
- *there exists $C_1 < \infty$ and a probability measure ν on Γ so that*

$$\nu\{\gamma \in \Gamma : \gamma \cap B \neq \emptyset\} \leq C_1 r^\beta \tag{3.3}$$

for each ball B in X of radius r.

Then $\beta \leq \alpha - 1$ and

$$\mathcal{C} \dim X \geq \frac{\alpha}{\alpha - \beta}. \tag{3.4}$$

As β ranges from 0 to $\alpha - 1$, the lower bound in (3.4) ranges from 1 to α. The "best" case is $\beta = \alpha - 1$, in which case $\mathcal{C} \dim X = \alpha$. The curve family Γ in Example 3.2 satisfies (3.3) with $\beta = \alpha - 1$.

The conclusion of Lemma 3.9 is nontrivial only when $\alpha/(\alpha - \beta) > \dim_T X$. We give one example which shows that this can occur even in the case $\beta < \alpha - 1$.

Example 3.10. Let $X \subset \mathbb{R}^2$ be the *Sierpinski carpet*, obtained from the unit square in $\mathbb{R}^2$ by repeated applications of the following operation: divide each of the current squares into 9 equal subsquares and remove the central subsquare. The space X is compact and α-regular, where $\alpha = \log 8/\log 3$. Let $E \subset \mathbb{R}$ denote the standard Cantor set and let ν be a β-regular probability measure on E, $\beta = \log 2/\log 3$. Since $E \times [0, 1] \subset X$, we may define a family of curves

$$\Gamma = \{\gamma_y : [0, 1] \to X | \gamma_y(t) = (y, t), y \in E\}.$$

For each $y \in E$, $\operatorname{diam} \gamma_y = 1$. Moreover, the measure ν on E induces a measure (which we continue to denote by ν) on Γ for which (3.3) is satisfied. Thus

$$\mathcal{C} \dim X \geq \frac{\log 8/\log 3}{\log 8/\log 3 - \log 2/\log 3} = \frac{3}{2}.$$

Note that $\dim_T X = 1$.

In the "best" case $\beta = \alpha - 1$, the result of Lemma 3.9 is contained in that of Theorem 3.4, provided that Γ contains a collection of rectifiable curves with positive ν-measure.

Proposition 3.11. *Let X be a compact α-regular space satisfying the hypotheses of Lemma 3.9 with $\beta = \alpha - 1$ and assume that $\nu(\Gamma') > 0$, where $\Gamma' \subset \Gamma$ denotes the collection of rectifiable curves. Then X has nontrivial α-modulus.*

Proof. By passing to a further subcollection if necessary, we may assume that the lengths of the curves in Γ' are uniformly bounded, say $\operatorname{length}(\gamma) \le M$ for all $\gamma \in \Gamma'$. For each such curve γ, let m_γ be the normalized arc length measure ds/M on γ. Choose $\lambda > 1$. If B is a ball with radius $r < \delta/(2\lambda)$, then no curve $\gamma \in \Gamma'$ can lie completely in λB. Thus $\gamma \cap B \neq \emptyset$ implies that $\operatorname{length}(\gamma \cap \lambda B) \ge (\lambda - 1)r$. Hence

$$\begin{aligned}
\int_{\{\gamma\in\Gamma':\gamma\cap B\neq\emptyset\}} & m_\gamma(\gamma\cap\lambda B)^{1-\alpha}\, d\nu(\gamma) \\
&= M^{\alpha-1}\int_{\{\gamma\in\Gamma':\gamma\cap B\neq\emptyset\}} \operatorname{length}(\gamma\cap\lambda B)^{1-\alpha}\, d\nu(\gamma) \\
&\le \left(\frac{M}{\lambda-1}\right)^{\alpha-1} r^{1-\alpha}\nu\{\gamma\in\Gamma' : \gamma\cap B\neq\emptyset\} \le C'
\end{aligned}$$

where C' depends only on M, λ, α and the constant C_1 in (3.3). By Proposition 2.9 of [9], $\operatorname{mod}_{\alpha,k,l,m}\Gamma' > 0$ for each $k \ge 1$, $l \ge 5\lambda k$ and $m \ge 1$. Choosing k, l, m appropriately, we conclude (by Proposition 3.8(ii)) that $\operatorname{Mod}_\alpha \Gamma' > 0$. □

We turn now to the proof of Theorem 3.4, which relies on the following simple observation.

Lemma 3.12. *Let X be uniformly perfect and let Γ be a family of subsets of X with $\operatorname{diam}\gamma \ge \delta > 0$ for each $\gamma \in \Gamma$. Let $\alpha > 0$. If $\operatorname{mod}_{\alpha,k,l,m}\Gamma > 0$ for some $l \ge k \ge 1$, $m \ge 1$, then $\dim X \ge \alpha$.*

Proof. Define a map $\varphi : \mathcal{B}(X) \to [0,\infty]$ by $\varphi(A) = \operatorname{diam} A/\delta$. By (3.1),

$$\Phi_{1,m}(\gamma) \ge \frac{1}{\delta}\mathcal{H}_1(\gamma) \ge \frac{1}{\delta}\operatorname{diam}\gamma \ge 1$$

for all $\gamma \in \Gamma$ and so by (3.2),

$$0 < \operatorname{mod}_{\alpha,k,l,m}\Gamma \le \tilde{\Phi}_{\alpha,k,l}(X) \le C\Phi_{\alpha,k}(X) \le C\mathcal{H}_\alpha(X)$$

where C depends only on α, δ, l, and a constant c_1 of uniform perfectness for X. Hence $\dim X \ge \alpha$. □

Lemma 3.13. *Let (X,μ) be a metric measure space, $p > 1$, and let Γ be a family of curves in X with $\operatorname{Mod}_p \Gamma > 0$. Then there exists $\delta > 0$ and a subfamily $\Gamma' \subset \Gamma$ with $\operatorname{Mod}_p \Gamma' > 0$ for which $\operatorname{diam}\gamma \ge \delta$ for all $\gamma \in \Gamma'$.*

Proof. For each $m \in \mathbb{N}$, let Γ_m denote the collection of those $\gamma \in \Gamma$ for which $\operatorname{diam}\gamma \ge 1/m$. Then $\Gamma = \bigcup_{m=1}^\infty \Gamma_m$. By a result of Ziemer [15, Lemma 2.3], $\operatorname{Mod}_p\Gamma = \sup_{m\ge1}\operatorname{Mod}_p\Gamma_m$ and so $\operatorname{Mod}_p\Gamma_m > 0$ for some $m \in \mathbb{N}$. □

Proof of Theorem 3.4. Let X be a compact α-regular space, $\alpha > 1$, and let Γ be a curve family in X with $\operatorname{Mod}_\alpha\Gamma > 0$. Let $f : X \to Y$ be an η-quasisymmetric map onto a (compact) space Y. By Lemma 3.13, we may assume that $\operatorname{diam}\gamma \ge \delta > 0$ for all $\gamma \in \Gamma$.

By Proposition 3.8(ii), for any $k, l, m \ge 1$ with $l \ge 10k$, we have

$$\operatorname{mod}_{\alpha,k,l,m}\Gamma > 0.$$

Choosing k, l, m large enough (depending on η), we can guarantee (by Proposition 3.8(i)) that there exist $k_1, l_1, m_1 \geq 1$ with

$$\operatorname{mod}_{\alpha,k_1,l_1,m_1} f\Gamma > 0.$$

Since X and Y are compact and $\operatorname{diam}\gamma \geq \delta > 0$ for all $\gamma \in \Gamma$, we have $\operatorname{diam} f\gamma \geq \delta'$ for all $\gamma \in \Gamma$, where $\delta' = \delta'(\eta, \delta) > 0$ [11, Theorem 2.5]. Finally, Y (as a quasisymmetric image of a uniformly perfect space) is also uniformly perfect, and so, by Lemma 3.12, $\dim Y \geq \alpha$. The proof is complete. □

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

E-mail address: `jttyson@math.lsa.umich.edu`

Current address: Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11794-3651

E-mail address: `tyson@math.sunysb.edu`