ON THE SET OF POINTS WITH A DENSE ORBIT

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Abstract. Under certain conditions on the topological space $X$ we prove that for every continuous map $f: X \to X$ the set of all points with a dense orbit has empty interior in $X$. This result implies a negative answer to two problems proposed by M. Barge and J. Kennedy.

1. Introduction

Let $X$ be a topological space. Given a subset $A$ of $X$ we shall denote by $Cl_X(A)$ and $Int_X(A)$ the closure and the interior of $A$ in $X$, respectively. If $U, V$ are subsets of $X \times X$ and $a \in X$, we define

$$U \circ V = \{(a, b) \in X \times X; (a, c) \in U \text{ and } (c, b) \in V \text{ for some } c \in X\}$$

and

$$U(a) = \{b \in X; (a, b) \in U\}.$$

Moreover, if $f: X \to X$ is a continuous map, the orbit of a point $x \in X$ under $f$ is the set

$$O_f(x) = \{f^j(x); j \geq 0\}.$$

Finally, we shall denote by $D_f$ the set of all points $x \in X$ whose orbit $O_f(x)$ is dense in $X$.

M. Barge and J. Kennedy [1] proposed the following problems (cf. Problems 5 and 6 on page 641 of [1]):

- Let $\{p_1, p_2, \ldots, p_n\}$ be a set of $n \geq 2$ distinct points in the sphere $S^2$. Is there a homeomorphism of $S^2 - \{p_1, p_2, \ldots, p_n\}$ such that every orbit of the homeomorphism is dense?
- Is there a homeomorphism of $\mathbb{R}^n$, $n \geq 3$, such that every orbit of the homeomorphism is dense?

Our goal is to answer both questions in the negative. More precisely, we shall see that if $X = S^2 - \{p_1, p_2, \ldots, p_n\}$ or $X = \mathbb{R}^n$ ($n \geq 1$), then any continuous map $f: X \to X$ has the property that $Int_X(D_f) = \emptyset$ (in particular, $D_f \neq X$). In fact, our results apply to a much larger class of spaces $X$ (cf. Theorem 1 and Corollary 6).
2. Main results

**Theorem 1.** Let $X$ be a locally compact Hausdorff space which is not compact and has no isolated points. Then, for any continuous map $f : X \to X$, we have $\text{Int}_X(D_f) = \emptyset$.

**Proof.** Let $Y$ be the one-point compactification of $X$ and let $\mathcal{U}$ be the set of all open symmetric vicinities of the unique uniform structure compatible with the topology of $Y$. Fix $z \in \text{Cl}_Y(X) - X$ and suppose that $\text{Int}_X(D_f) \neq \emptyset$. Then, there are $x_0 \in X$ and $V \in \mathcal{U}$ such that

$$\text{Cl}_Y(V(x_0)) \subset D_f.$$  

For each $U \in \mathcal{U}$, choose an integer $r_U \geq 1$ such that $f^{r_U}(x_0) \in U(z)$. Let $\alpha_U \leq r_U$ be the greatest integer such that $f^{\alpha_U}(x_0) \in V(x_0)$ and let $\beta_U > r_U$ be the smallest integer such that $f^{\beta_U}(x_0) \in V(x_0)$. Put $y_U = f^{\alpha_U}(x_0)$. Then we have the following properties:

$$y_U, f^{\beta_U - \alpha_U}(y_U) \in V(x_0),$$

$$f(y_U), f^2(y_U), \ldots, f^{\beta_U - \alpha_U - 1}(y_U) \notin V(x_0)$$

and

$$f^{r_U - \alpha_U}(y_U) \in U(z).$$

Let $y_0 \in Y$ be a cluster point of the net $(y_U)_{U \in \mathcal{U}}$. Since $y_0 \in \text{Cl}_Y(V(x_0)) \subset D_f$, there is a smallest integer $M \geq 1$ such that

$$f^M(y_0) \in V(x_0).$$

Let $Z \in \mathcal{U}$ and $V' \in \mathcal{U}$ be such that

$$(f^M(y_0), x_0) \in V' \quad \text{and} \quad Z \circ V' \subset V.$$

For each $U \in \mathcal{U}$, let $N_U$ be a neighborhood of $y_0$ in $Y$ such that

$$(f^n(y), f^n(y_0)) \in Z \cap U \quad \text{for } n = 0, \ldots, M, \quad \text{whenever } y \in X \cap N_U.$$

Since $y_0$ is a cluster point of $(y_U)_{U \in \mathcal{U}}$, there is a $W_U \in \mathcal{U}$, $W_U \subset U$, such that $y_{W_U} \in N_U$.

Thus, $f^M(y_{W_U}) \in V(x_0)$. Since $\beta_{W_U} - \alpha_{W_U} \leq M$, we obtain

$$f^{r_{W_U} - \alpha_{W_U}}(y_0), z \in U \circ U.$$

Therefore, $z \in \text{Cl}_Y((y_0, f(y_0), \ldots, f^M(y_0)))$, and so

$$z = f^j(y_0) \quad \text{for some } 0 \leq j \leq M.$$

But this is a contradiction, since $z \notin X$.

As an immediate consequence of Theorem 1 we obtain the following negative answers to the two problems mentioned in the introduction:

**Corollary 2.** For any integer $n \geq 1$ and any continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$, we have $\text{Int}_{\mathbb{R}^n}(D_f) = \emptyset$.

**Corollary 3.** Let $X = S^2 - \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ $(n \geq 1)$ are distinct points of $S^2$. Then, for any continuous map $f : X \to X$, we have $\text{Int}_X(D_f) = \emptyset$. 
Remark 4. The conclusion of Corollary 2 is not true if we consider infinite-dimensional Banach spaces \( \mathbb{X} \) in place of \( \mathbb{R}^n \). Indeed, in certain infinite-dimensional Banach spaces \( \mathbb{X} \), Read [2] proved the existence of a continuous map \( f : \mathbb{X} \to \mathbb{X} \) such that \( D_f = \mathbb{X} - \{0\} \) (his map is even linear). This also shows that we cannot omit the local compactness hypothesis in Theorem 1.

3. FURTHER RESULTS

The argument used in the proof of Theorem 1 can also be applied to establish the following:

**Theorem 5.** Let \( \mathbb{X} \) be a compact Hausdorff space without isolated points. Then, for any continuous map \( f : \mathbb{X} \to \mathbb{X} \), either \( D_f = \mathbb{X} \) or \( \text{Int}_\mathbb{X}(D_f) = \emptyset \).

*Proof.* Suppose \( D_f \neq \mathbb{X} \) and \( \text{Int}_\mathbb{X}(D_f) \neq \emptyset \). Fix an \( a \in \mathbb{X} - D_f \). By arguing as in the proof of Theorem 1 (with \( z = a \)), we conclude that for some \( y_0 \in D_f \) and some \( j \geq 0 \),

\[
    f^j(y_0) = z = a.
\]

But this is a contradiction, since \( y_0 \in D_f \) and \( a \notin D_f \).

**Corollary 6.** Let \( \mathbb{X} \) be a compact convex set in a Hausdorff locally convex space and assume that \( \mathbb{X} \) is not a singleton. Then, for any continuous map \( f : \mathbb{X} \to \mathbb{X} \), we have \( \text{Int}_\mathbb{X}(D_f) = \emptyset \).

*Proof.* By the Schauder-Tychonoff fixed point theorem, \( f \) has a fixed point in \( \mathbb{X} \). So, we cannot have \( D_f = \mathbb{X} \).

Remark 7. (a) The possibility \( D_f = \mathbb{X} \) can happen in Theorem 5. For instance, let \( \mathbb{X} \) be the unit circle and let \( f : \mathbb{X} \to \mathbb{X} \) be a rotation by an irrational number.

(b) We cannot omit the hypothesis that \( \mathbb{X} \) has no isolated points in Theorem 5. For instance, consider \( \mathbb{X} = \{0, 1\} \) and define \( f : \mathbb{X} \to \mathbb{X} \) by \( f(0) = 0 \) and \( f(1) = 0 \). Then \( D_f = \{1\} \), which is not \( \mathbb{X} \) and does not have empty interior.

(c) In view of Read’s example [2], we cannot omit the compactness hypothesis in Theorem 5.

REFERENCES


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