FINSLER METRICS AND ACTION POTENTIALS

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Abstract. We study the behavior of Mañé’s action potential $\Phi_k$ associated to a convex superlinear Lagrangian, for $k$ bigger than the critical value $c(L)$. We also prove that the action potential can be written as $\Phi_k(x, y) = D_F(x, y) + f(y) - f(x)$ where $f$ is a smooth function and $D_F$ is the distance function associated to a Finsler metric.

1. Introduction

Let $M$ be a closed Riemannian manifold with Riemannian metric $\langle v, v \rangle$. Consider the mechanical Lagrangian $L : TM \to \mathbb{R}$,

$$ (x, v) \mapsto \frac{1}{2} \langle v, v \rangle_x - U(x) $$

where $U(x)$ is a differentiable function on $M$ called the potential.

It is well known that, on a fixed level of energy $e$, bigger than the maximum of $U$ the Lagrangian is conjugate to the geodesic flow with metric $2(e - U(x))\langle v, v \rangle$. Moreover the reduced action of the Lagrangian is the distance for this metric. Either or both of these statements are known as the Maupertuis principle. See the books [1], [2] or [5].

Consider now a general convex superlinear Lagrangian $L : TM \to \mathbb{R}$. This means that $L$ restricted to each $T_x M$ has positive definite Hessian and

$$ \lim_{|v| \to \infty} \frac{L(x, v)}{|v|} = \infty, $$

uniformly on $x \in M$.

It was proven in [4] that for large energy values the Lagrangian flow is conjugate to the flow of a Finsler metric. See below for the precise statement. In Theorem 1 we prove a generalization of the other statement of the Maupertuis principle. This was motivated by discussions with R. Montgomery about the presentation in [5], which also motivated Theorem 2.

Let $H : T^* M \to \mathbb{R}$ be the Hamiltonian associated to $L$ and let $\mathcal{L} : TM \to T^* M$ be the Legendre transform $(x, v) \mapsto \partial L/\partial v(x, v)$. Since $M$ is compact, the extremals of $\mathcal{L}$ give rise to a complete flow $\varphi_t : TM \to TM$ called the Euler-Lagrange flow of

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the Lagrangian. Using the Legendre transform we can push forward \( \varphi_t \) to obtain another flow \( \varphi_t^* \) which is the Hamiltonian flow of \( H \) with respect to the canonical symplectic structure of \( T^*M \). Recall that the energy \( E : TM \to \mathbb{R} \) is defined by

\[
E(x, v) = \frac{\partial L}{\partial v}(x, v) \cdot v - L(x, v).
\]

Since \( L \) is autonomous, \( E \) is a first integral of the flow \( \varphi_t \).

Recall that the action of the Lagrangian \( L \) on an absolutely continuous curve \( \gamma : [a, b] \to M \) is defined by

\[
A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt.
\]

Given two points \( x \) and \( y \) in \( M \) and \( T > 0 \) denote by \( C_T(x, y) \) the set of absolutely continuous curves \( \gamma : [0, T] \to M \), with \( \gamma(0) = x \) and \( \gamma(T) = y \). For each \( k \in \mathbb{R} \) we define the action potential \( \Phi_k : M \times M \to \mathbb{R} \) by

\[
\Phi_k(x, y) = \inf \{ A_{L+k}(\gamma) : \gamma \in \bigcup_{T>0} C_T(x, y) \}.
\]

The critical value of \( L \), which was introduced by Mañé in \cite{Mañé}, is the real number \( c(L) \) defined as the infimum of \( k \in \mathbb{R} \) such that for some \( x \in M \), \( \Phi_k(x, x) > -\infty \). For \( k \geq c(L) \), we have that \( \Phi_k(x, y) > -\infty \) for every \( x, y \) and it is a Lipschitz function that satisfies the triangle inequality.

For any \( k > c(L) \) the flow on the energy level \( k \) is conjugate to the geodesic flow of an appropriately chosen Finsler metric on \( M \) (see \cite{Finsler}).

Given a Finsler metric \( \sqrt{F} \) and an absolutely continuous curve \( \gamma \) we can define its Finsler length as

\[
l_F(\gamma) = \int \sqrt{F(\dot{\gamma})}.
\]

Observe that since the Finsler metric is homogeneous of degree one, the definition does not depend on the parameterization of the curve. Finally we define the Finsler distance as

\[
D_F(x, y) = \inf \{ l_F(\gamma) \}
\]

where the infimum is taken over all absolutely continuous curves joining \( x \) and \( y \).

**Theorem 1.** If \( k \) is bigger than the critical value, then there exist a Finsler metric \( \sqrt{F} \) and a \( C^\infty \) real valued function \( f \) on \( M \) such that \( \Phi_k(x, y) = D_F(x, y) + f(y) - f(x) \). Moreover if \( k \) is bigger than \( -\inf L \), then we can choose \( f = 0 \).

As a consequence of Theorem 1 we have that there is a neighborhood \( V \) of the diagonal \( \Delta \) in \( M \times M \), such that \( \Phi_k \) is differentiable in \( V \setminus \Delta \).

For \( x, y \) fixed and \( T > 0 \) define

\[
S(T) = \inf \{ A_L(\gamma) : \gamma \in C_T(x, y) \}.
\]

It is easily shown that \( S(T) \) is continuous. Although \( S(T) \) is not necessarily convex, its Legendre transform:

\[
S^*(e) = \max_{T>0}(eT - S(T))
\]
is a well defined convex function and coincides with the Legendre transform of the convex hull $\overline{S}$ of $S$. Notice that

$$\Phi_k(x, y) = -S^*(-k)$$

and so the domain of $S^*$ is $\text{dom} S^* = (-\infty, -c(L)]$. It follows from the definition of the action potential that $g(k) = \Phi_k(x, y)$ is nondecreasing and so is $S^*$.

**Theorem 2.** For all $x, y$ in $M$ we have that:

(a) $g$ grows slower than any linear function; that is,

$$\lim_{k \to \infty} \frac{g(k)}{k} = 0.$$

(b) The right derivative of $g$ at $c(L)$ is infinite.

(c) $\lim_{T \to \infty} S(T)/T = -c(L)$.

2. Proofs

**Proof of Theorem 2.** It is well known that if $f$ is a convex function of a real variable, then

1. If $x \in \text{int}(\text{dom} f)$, then both one side derivatives exist and $f'_-(x) \leq f'_+(x)$.
2. If $x \in \text{dom} f$ is a boundary point, then the corresponding one side derivative exists.
3. If $x < y$, $f'_+(x) \leq f'_-(y)$.

Define

$$\text{rang } \partial f = \bigcup_{x \in \text{dom} f} [f'_-(x), f'_+(x)].$$

It is proved in [1], Section 24, that

$$\text{int}(\text{dom} f^*) \subset \text{rang } \partial f \subset \text{dom} f^*.$$

Therefore

$$\text{rang } \partial S^* = \text{dom } S^{**} = \text{dom } \overline{S} = (0, \infty).$$

Thus

$$\lim_{e \to -\infty} \frac{S^*(e)}{e} = 0$$

and

$$S^*_-(c(L)) = \lim_{e \to -c(L)} S^*_-(e) = \infty.$$

From equation (11), items (a) and (b) follow.

By the same kind of arguments $\lim_{T \to \infty} \overline{S}(T)/T = -c(L)$, and then

$$-c(L) \leq \liminf_{T \to \infty} \frac{S(T)}{T}.$$

To prove the other inequality, let $\mu$ be an ergodic minimizing probability, that is, an invariant ergodic probability for the lagrangian flow $\varphi_t$ such that

$$m := \int L \, d\mu \leq \int L \, d\nu.$$
for any invariant probability $\nu$. Mather proved that such measures exist (see [7]). Let $\theta \in TM$ be a regular point for $\mu$, such that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T L(\varphi_t(\theta))dt = m.$$ 

Let $\pi : TM \to M$ be the natural projection. Comparing with the curve $\gamma$ obtained by joining $x$ and $\pi(\theta)$ with a short curve, then following the curve $\pi\varphi_t(\theta)$ and then joining $\pi\varphi_T(\theta)$ and $y$ with a short curve, we have that for any given $\varepsilon > 0$ and $T$ large enough

$$S(T) \leq (m + \varepsilon)T + O(1).$$

So

$$\limsup_{T \to \infty} \frac{S(T)}{T} \leq (m + \varepsilon).$$

Item (c) now follows from the fact due to Mañé [6, 3] that $m = c(L)$.

Proof of Theorem 1. We begin with the last statement. Note that $L + k$ is bigger than zero if and only if $H(x, 0) < k$. Indeed

$$H(x, p) = \max_{v \in T_xM} (pv - L(x, v));$$

then

$$H(x, 0) = \max_{v \in T_xM} (-L(x, v)) = -\min_{v \in T_xM} (L(x, v)).$$

So if $k$ is bigger than $-\inf L$, then $H^{-1}(-\infty, k)$ contains the zero section of $T^*M$.

Now define a new Hamiltonian $G$ on $T^*M$ minus the zero section such that $G$ takes the value $\mu$ on $H^{-1}(k)$ and such that $G(x, \lambda p) = \lambda^2G(x, p)$ for all positive $\lambda$. Since $G$ is positively homogeneous of degree two and convex in $p$, it follows that the Legendre transform $F$ associated to $G$ is the square of a Finsler metric.

Since by definition $G^{-1}(\mu) = H^{-1}(k)$, it follows that the Hamiltonian flows of $G$ and $H$ coincide up to reparameterization on the energy level $G^{-1}(\mu) = H^{-1}(k)$ and therefore the Euler-Lagrange solutions of $L$ with energy $k$ are reparameterizations of geodesics of $\sqrt{F}$.

We claim that for an appropriate choice of $\mu$ and if $E(x, v) = k$, then

$$\sqrt{F(x, v)} = L + k.$$ 

It is proved in [6, 8] that for $k$ greater than the critical value and for any $x, y$ in $M$ there exists $\gamma$ such that $A_{L+k}(\gamma) = \Phi_k(x, y)$. Moreover $\gamma$ is a solution of the Euler-Lagrange equation and has energy $k$. Also, if $k > c(L)$, every curve can be reparameterized to have energy $k$ and the Finsler length does not depend on the reparameterization. By the definitions of both $DF$ and $\Phi_k$, we may restrict ourselves to curves with energy $k$ and Theorem follows in this case.

Proof of the claim. Since $G$ is homogeneous of degree 2 it follows from Euler’s formula that $F$ and $G$ take the same value at Legendre related points.

Define $\lambda(x, p)$ such that $H(x, \frac{p}{\lambda}) = k$; then $G(x, p) = \mu\lambda^2(x, p)$. We have

$$\frac{\partial H}{\partial p}(x, \frac{p}{\lambda})\lambda^{-1} - \frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \cdot p \lambda^{-2} \frac{\partial \lambda}{\partial p} = 0$$ 

(2)
and

$$\frac{\partial G}{\partial p} = 2\mu \lambda \frac{\partial \lambda}{\partial p}.$$ 

Multiplying (2) by $2\mu \lambda^3$ we then get

(3) $$\frac{\partial H}{\partial p}(x, \frac{p}{\lambda}) \cdot p \frac{\partial G}{\partial p}(x, \frac{p}{\lambda}) = 2G(x, p)\frac{\partial H}{\partial p}(x, \frac{p}{\lambda}).$$

Suppose now that $E(x, v) = k$ and let $P(x, v) = \partial L/\partial v$; then by definition we have

$$\lambda(x, P(x, v)) = 1,$$
$$G(x, P(x, v)) = \mu,$$
$$\frac{\partial H}{\partial p}(x, P(x, v)) = v,$$

and so

(4) $$\frac{\partial H}{\partial p}(x, P(x, v)) \cdot P(x, v) = v \frac{\partial L}{\partial v}$$
$$= L + k$$
$$> 0.$$ 

Hence from (3) we have

$$\frac{\partial G}{\partial p}(x, P(x, v)) = \frac{2v}{v \cdot P(x, v)}.$$ 

Since $\partial G/\partial p$ is homogeneous of degree one and from (4) $v \cdot P(x, v)$ is positive, we obtain

$$\frac{\partial G}{\partial p}(x, \frac{1}{2}v \cdot P(x, v)P(x, v)) = v.$$ 

So $v$ is related to $\frac{1}{2}v \cdot P(x, v)P(x, v)$ with respect to the Legendre transform of $F$. Hence

$$F(x, v) = G(x, \frac{1}{2}v \cdot P(x, v)P(x, v))$$
$$= \frac{(v \cdot P(x, v))^2}{4}G(x, P(x, v))$$
$$= \frac{(v \cdot P(x, v))^2}{4} \mu.$$ 

So if $\mu = 4$,

$$\sqrt{F(x, v)} = v \cdot \frac{\partial L}{\partial v}.$$ 

Now let $k$ be bigger than $c(L)$. Then by a corollary in [4] there exists a $C^\infty$ real valued function $f$ on $M$, such that $H(x, df_x) < k$. Define as in [4] $H_{df}(x, p) = H(x, p + df_x)$. The Hamiltonian flows are conjugate by $\psi(x, p) = (x, p - df_x)$. The
Legendre transformation $L_{df}$ of $H_{df}$ is
\[
L_{df}(x, v) = \max_{p \in T^*_x M} (pv - H_{df}(x, p))
= \max_{p \in T^*_x M} (pv - H(x, p + df_x))
= \max_{p \in T^*_x M} ((p - df_x)v - H(x, p))
= L(x, v) - df_x v.
\]

It turns out that
\[
E(L_{df}) = E(L),
\]
\[
c(L_{df}) = c(L),
\]
\[
\Phi_k(L_{df})(x, y) = \Phi_k(L)(x, y) - f(y) + f(x).
\]
So as the zero section is contained in $H_{df}^{-1}(-\infty, k)$, $L_{df} + k$ is positive and there is a Finsler metric such that
\[
\Phi_k(L_{df})(x, y) = D_F(x, y).
\]
So
\[
\Phi_k(L)(x, y) = D_F(x, y) + f(y) - f(x).
\]

References


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