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ORTHOGONALITY OF THE RANGE AND THE KERNEL OF SOME ELEMENTARY OPERATORS

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ABSTRACT. We prove the orthogonality of the range and the kernel of an important class of elementary operators with respect to the unitarily invariant norms associated with norm ideals of operators. This class consists of those mappings $E : B(H) \to B(H)$, $E(X) = AXB + CXD$, where $B(H)$ is the algebra of all bounded Hilbert space operators, and A, B, C, D are normal operators, such that $AC = CA$, $BD = DB$ and $\ker A \cap \ker C = \ker B \cap \ker D = \{0\}$. Also we establish that this class is, in a certain sense, the widest class for which such an orthogonality result is valid. Some other related results are also given.

1. Introduction

Following [1] we first define orthogonality in Banach spaces. If X is a complex Banach space, and if $x, y \in X$, we say that x is orthogonal to y if for all λ, $\mu \in \mathbb{C}$ we have

$$||\lambda x + \mu y|| \geq ||\mu y|| \,. \tag{1}$$

Dividing (1) by $|\lambda|$ (or $|\mu|$) we see that nothing is lost if one of the scalars λ, μ is omitted in (1).

If X_1, X_2 are (linear) subspaces of X, we say that X_1 is orthogonal to X_2 if for all $x \in X_1$, $y \in X_2$ we have $||x + y|| \geq ||y||$.

Note that if x is orthogonal to y, then y need not be orthogonal to x. Indeed, in $\mathbb{C}^2$ space with the max-norm $||(z, w)|| = \max(|z|, |w|)$, consider the vectors $(-1, 0)$ and $(1, 1)$.

Remark 1. If X is a Hilbert space, from (1) follows $\langle x, y\rangle = 0$, i.e. orthogonality in the usual sense.

Next we define unitarily invariant norms (u.i. norms in further text). Let H denote a separable Hilbert space, let $B(H)$ denote the algebra of all bounded linear operators on H, and let $\mathcal{C}_\infty$ denote the ideal of all compact operators. For a compact operator X the sequence of singular values $s_j(X)$ is defined as the sequence of eigenvalues of the operator $(X^*X)^{1/2}$ arranged in nonincreasing order, i.e. $s_j(X) =$

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$\lambda_j(X^*X)^{1/2}$. In a similar way, with few modifications, the singular values of an arbitrary bounded operator can be defined.

A unitarily invariant norm is any norm $|||\quad|||$ defined on some two-sided ideal of $B(H)$ (including $B(H)$ itself) which satisfies the two following conditions. For unitary operators U, $V \in B(H)$ the equality $|||UXV||| = |||X|||$ holds, and $|||X||| = s_1(X)$ for all rank one operators X. It is proved that any u.i. norm depends only on the sequence of singular values. Also, it is known that the maximal ideal, on which $|||\quad|||$ has sense, is a Banach space with respect to that u.i. norm.

Among all u.i. norms there are few important special cases. The first is the Schatten p-norm ($p \geq 1$) defined by $||X||_p = (\sum_{j=1}^{+\infty} s_j(X)^p)^{1/p}$ on the set $\mathcal{C}_p = \{X \in B(H) | \; ||X||_p < +\infty\}$. For $p = 1$ ($p = 2$) this norm is known as the nuclear norm (Hilbert-Schmidt norm) and the corresponding ideal is known as the ideal of nuclear (Hilbert-Schmidt) operators. The ideal $\mathcal{C}_2$ is also interesting for another reason. Namely, it is a Hilbert space with respect to the $||\quad||_2$ norm. The other important special case is the set of so-called Ky Fan norms $||X||_{(k)} = \sum_{j=1}^{k} s_j(X)$. The well-known Ky Fan dominance property asserts that the condition $||X||_{(k)} \leq ||Y||_{(k)}$ for all $k \geq 1$ is necessary and sufficient for the validity of the inequality $|||X||| \leq |||Y|||$ in all u.i. norms.

For further details the reader is referred to [4].

Anderson [1] considered the operator $\delta_{A,B} : B(H) \to B(H)$ defined by $\delta_{A,B}(X) = AX - XB$. If A and B are normal, then for all X,$S \in B(H)$ he proved the implication

$$AS = SB \Longrightarrow ||AX - XB + S|| \geq ||S||$$

which means that the range of $\delta_{A,B}$ is orthogonal to its kernel.

Kittaneh [2] extended this result to an arbitrary u.i. norm, and proved the following theorem.

Theorem AK. *If the operators A,$B \in B(H)$ are normal, then for all X, $S \in B(H)$ we have*

$$S \in \ker \delta_{A,B} \Longrightarrow |||\delta_{A,B}(X) + S||| \geq |||S||| .$$

In this note we shall extend the above theorem. Namely, among other things, we shall prove the implication

$$S \in \ker E \Longrightarrow |||E(X) + S||| \geq |||S|||$$

where $E : B(H) \to B(H)$ is defined by $E(X) = AXB + CXD$, where $AC = CA$, $BD = DB$, and $A^*A + C^*C > 0$, $B^*B + D^*D > 0$, i.e. $\ker A \cap \ker C = \ker B \cap \ker D = \{0\}$.

2. Some auxilliary results

We first prove and list some results which will be needed later.

Lemma 1. *If inequality* (1) *holds, then $||\lambda x + \mu y|| \geq 2^{-1} ||\lambda x||$, for all $\lambda, \mu \in \mathbb{C}$, and if the last inequality holds, then $||\lambda x + \mu y|| \geq 3^{-1} ||\mu y||$, for all $\lambda, \mu \in \mathbb{C}$.*

Proof. We have $||\lambda x|| = ||\lambda x + \mu y - \mu y|| \leq ||\lambda x + \mu y|| + ||\mu y|| \leq 2 ||\lambda x + \mu y||$, and also $||\mu y|| = ||\lambda x + \mu y - \lambda x|| \leq ||\lambda x + \mu y|| + ||\lambda x|| \leq 3 ||\lambda x + \mu y||$.

Lemma 2. *Let X, $Y \in B(H)$, $Z \in B(H_1)$, and $|||X||| \leq |||Y|||$ in every u.i. norm. Then $|||X \oplus Z||| \leq |||Y \oplus Z|||$ in every u.i. norm.*

Proof. It is enough to prove the inequality $||X \oplus Z||_n \leq ||Y \oplus Z||_n$, where $||A||_n$ denotes $\sum_{j=1}^n s_j(A)$. But, for some $0 \leq k \leq n$ we have $||X \oplus Z||_n = ||X||_k + ||Z||_{n-k} \leq ||Y||_k + ||Z||_{n-k} \leq ||Y \oplus Z||_n$, in view of the nonincreasing order of the sequence s_j.

Lemma K. *If $T \in \mathcal{C}_\infty$ has the representation $T = [T_{ij}]_{i,j=1}^2$ with respect to some orthogonal decomposition of a Hilbert space H, then:*

a) $2^{p-2} \sum_{i,j=1}^2 ||T_{ij}||_p^p \leq ||T||_p^p \leq \sum_{i,j=1}^2 ||T_{ij}||_p^p$, *for* $1 \leq p \leq 2$,

b) $\sum_{i,j=1}^2 ||T_{ij}||_p^p \leq ||T||_p^p \leq 2^{p-2} \sum_{i,j=1}^2 ||T_{ij}||_p^p$, *for* $2 \leq p < +\infty$,

where $|| \quad ||_p$ *denotes the* $\mathcal{C}_p$*-norm.*

Proof. The proof of this lemma is based on Clarckson-McCarthy inequalities, and can be found in [3].

Lemma GK. *Let $\{P_j\}_{j=1}^\omega$ be a family of mutually orthogonal orthoprojectors, let $\mathcal{J}$ be a two-sided ideal contained in $\mathcal{C}_\infty$, and let $||| \quad |||_{\mathcal{J}}$ be its norm. Then $A \in \mathcal{J}$ implies $\sum_{j=1}^\omega P_j A P_j \in \mathcal{J}$ and $|||A|||_{\mathcal{J}} \geq ||| \sum_{j=1}^\omega P_j A P_j |||_{\mathcal{J}}$.*

Proof. This lemma is in fact Theorem III 4.2. from [4].

Remark 2. This statement remains valid for u.i. norms equivalent to the usual one on $B(H)$ which follows from the fact that the proof of Theorem III 4.2. is based on the Ky-Fan dominance property.

Lemma 3. *Let a bounded operator A be the weak limit of operators A_n and let $\sup_n |||A_n||| < +\infty$, for some u.i. norm $||| \quad |||$. Then $|||A||| \leq \sup_n |||A_n|||$.*

Proof. This result can be obtained from the proof of Lemma III 5.1. from [4], but for the convenience of the reader we shall give the proof.

Let $T_{n,k} = \sum_{j=1}^k s_j(A_n) \langle \quad, \phi_j^{(n)} \rangle \psi_j^{(n)}$ be the partial sum of the Schmidt expansion of operators A_n. We choose the sequence of indices n_i such that $s_j(A_{n_i})$ converges to s_j, and $\phi_j^{(n_i)}$, $\psi_j^{(n_i)}$ converge weakly to ϕ_j, ψ_j. Obviously $T_k = \sum_{j=1}^k s_j \langle \quad, \phi_j \rangle \psi_j$ is the weak limit of $T_{n_i,k}$. It is well known (see [4], p. 47) that we may choose a unitary operator U and an orthonormal system $\{f_j\}_{j=1}^k$ such that $\langle UAf_j, f_j \rangle = s_j(A)$. Then $\left| \sum_{j=1}^k \langle UA_{n_i} f_j, f_j \rangle \right| \leq \sum_{j=1}^k s_j(A_{n_i}) \leq \sup_m \sum_{j=1}^k s_j(A_m)$. Passing to the limit $n_i \to \infty$ we obtain the result.

3. Main results

Theorem 1. *Let A, $B \in B(H)$ be normal operators, such that $AB = BA$, and let $E(X) = AXB - BXA$. Furthermore, suppose that*

$$A^*A + B^*B > 0. \tag{2}$$

If $S \in \ker E$, then

$$|||E(X) + S||| \geq |||S|||. \tag{3}$$

Proof. Let us start with the case when $B^{-1} \in B(H)$. The assumptions $AB = BA$, $ASB = BSA$ clearly imply $AB^{-1}S = SB^{-1}A$. Hence, applying Theorem AK to the operators AB^{-1}, $B^{-1}A$, BXB and S we get

$$|||AXB - BXA + S||| = \left|\left|\left| AB^{-1}BXB - BXBB^{-1}A + S \right|\right|\right| \geq |||S|||.$$

Consider then the case when B is injective, i.e. $\ker B = \{0\}$. Let $\Delta_n = \{\lambda \in \mathbb{C} | |\lambda| \le 1/n\}$ and let $E_B(\Delta_n)$ be the corresponding spectral projector. Putting $P_n = I - E_B(\Delta_n)$ we have that subspace $P_n H$ reduces both operators A and B (since they commute and are normal). Hence, we take the following matrix representations of A, B, S, X with respect to the decomposition $H = (I - P_n)H \oplus P_n H$:

$$A = \begin{bmatrix} A_0^{(n)} & 0 \\ 0 & A_1^{(n)} \end{bmatrix}; \quad B = \begin{bmatrix} B_0^{(n)} & 0 \\ 0 & B_1^{(n)} \end{bmatrix};$$

$$S = \begin{bmatrix} S_{11}^{(n)} & S_{12}^{(n)} \\ S_{21}^{(n)} & S_{22}^{(n)} \end{bmatrix}; \quad X = \begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix}.$$

Operator $B_1^{(n)}$ acting on $P_n H$ is, as it is easily seen, invertible. Thus according to the previous case we have

$$\begin{aligned} (4) \qquad |||AXB - BXA + S||| &\ge |||P_n(AXB - BXA + S)P_n||| \\ &= \left|\left|\left| A_1^{(n)} X_{22}^{(n)} B_1^{(n)} - B_1^{(n)} X_{22}^{(n)} A_1^{(n)} + S_{22}^{(n)} \right|\right|\right| \\ &\ge \left|\left|\left| S_{22}^{(n)} \right|\right|\right| = |||P_n S P_n|||. \end{aligned}$$

Now, the sequence P_n strongly converges to $I - E_B(\{0\})$, i.e. to I, since B is injective, and hence $P_n S P_n \to S$ weakly (even strongly). From (4) we get $\sup_n |||P_n S P_n||| \le |||AXB - BXA + S||| < +\infty$. Applying Lemma 3 we obtain the required inequality. This ends the proof in the case when B is injective.

Assume now that $\ker A \cap \ker B = \{0\}$. Then $\ker B$ clearly reduces A and $P_{\ker B} A P_{\ker B}$ is injective. We split the space H into the orthogonal sum $H = \ker B \oplus H_0$ ($H_0 = H \ominus \ker B$). Let A, B, S, X have the following representation (with respect to the above decomposition):

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix};$$
$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}; \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Operators A_1, B_2 are injective, and (in view of their normality) their ranges are dense in the subspaces they act on. A straightforward calculation gives:

$$AXB - BXA = \begin{bmatrix} 0 & A_1 X_{12} B_2 \\ -B_2 X_{21} A_1 & A_2 X_{22} B_2 - B_2 X_{22} A_2 \end{bmatrix}.$$

The same block structure remains valid for $ASB - BSA$. Now, if $ASB - BSA = 0$, then $A_2 S_{22} B_2 = B_2 S_{22} A_2$, and $B_2 S_{21} A_1 = 0$, $A_1 S_{12} B_2 = 0$, i.e. $S_{21} = 0$; $S_{12} = 0$, since A_1, B_2 are injective and their ranges are dense. Thus

$$AXB - BXA + S = \begin{bmatrix} S_{11} & A_1 X_{12} B_2 \\ -B_2 X_{21} A_1 & A_2 X_{22} B_2 - B_2 X_{22} A_2 + S_{22} \end{bmatrix}.$$

However, since B_2 is injective, we have already proved that

$$|||A_2 X_{22} B_2 - B_2 X_{22} A_2 + S_{22}||| \ge |||S_{22}|||,$$

and applying Lemma GK with orthoprojectors $P_{\ker B}$, P_{H_0}, and Lemma 2 we have

$$\begin{aligned}|||AXB - BXA + S||| &\geq \left|\left|\left|\begin{bmatrix} S_{11} & 0 \\ 0 & A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22}\end{bmatrix}\right|\right|\right| \\ &\geq \left|\left|\left|\begin{bmatrix} S_{11} & 0 \\ 0 & S_{22}\end{bmatrix}\right|\right|\right| = |||S|||\,.\end{aligned}$$

The theorem is proved.

Even if the condition (2) is suppressed, we can get an inequality of the form

$$|||E(X) + S||| \geq c\,|||S||| \tag{5}$$

but the constant c can effectively be less than 1.

Example. Let α, β, $\gamma \in \mathbb{R}$, $\alpha > 0$, and let the operators A, X and S be defined on the Hilbert space $\mathbb{C}^4$ by matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}; \quad X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 1 & \gamma & -\gamma \\ 1 & 1 & -\gamma & \gamma \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix},$$

and consider the mapping $E(X) = AXA^* - A^*XA$. Clearly $E(S) = 0$ and

$$E(X) + S = \begin{bmatrix} 1 & 1 & \gamma & -\gamma \\ 1 & 1 & -\gamma & \gamma \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \alpha \end{bmatrix}.$$

By routine calculation we see that the singular values of S are 0, 2, α, $\sqrt{\alpha^2 + 4\gamma^2}$, whereas the singular values of $E(X) + S$ are 0, 2, $|\alpha + \beta|$, $\sqrt{(\alpha - \beta)^2 + 4\gamma^2}$. For $\beta = \alpha/2$, $\gamma = \sqrt{\alpha\beta}$, $\alpha \geq 4/3$ we get $||S|| = \alpha\sqrt{3}$, and $||E(X) + S|| = 3\alpha/2$, which means that the constant c in (5) cannot be greater than $\sqrt{3}/2$ in the case of the usual operator norm.

Again, for $\beta = -\alpha$, $\gamma = \alpha\sqrt{2}$, we get $||S||_1 = 2 + 4\alpha$, $||E(X) + S||_1 = 2 + \alpha 2\sqrt{3}$, implying that $||S||_1^{-1}\,||E(X) + S||_1 \to \sqrt{3}/2$ as $\alpha \to +\infty$. Hence, the constant c in (5) also cannot be greater than $\sqrt{3}/2$ in the nuclear norm.

We now prove the following result.

Theorem 2. *Let A, $B \in B(H)$ be normal operators such that $AB = BA$ and let $E(X) = AXB - BXA$. If $S \in \ker E$, then*

$$|||E(X) + S||| \geq 3^{-1}\,|||S||| \tag{6}$$

and

$$||E(X) + S||_p \geq 2^{-|1-2/p|}\,||S||_p \tag{7}$$

where $||\quad||_p$ is the $\mathcal{C}_p$-norm.

In particular, in the Hilbert-Schmidt-norm we have

$$||E(X) + S||_2^2 = ||S||_2^2 + ||E(X)||_2^2\,. \tag{8}$$

Proof. Let $H = H_1 \oplus H_2$, where $H_1 = \ker A \cap \ker B$, $H_2 = H \ominus H_1$, and let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}; \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}; \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

be the corresponding representations. We note that $\ker A_2 \cap \ker B_2 = \{0\}$ in H_2. Applying Lemma 1 and Theorem 1, we get

$$\begin{aligned} |||AXB - BXA + S||| &= \left|\left|\left| \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22} \end{bmatrix} \right|\right|\right| \\ &\geq |||A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22}||| \\ &\geq 2^{-1}\, |||A_2X_{22}B_2 - B_2X_{22}A_2||| \\ &= 2^{-1}\, |||AXB - BXA|||\,. \end{aligned}$$

Another application of Lemma 1 proves (6).

In order to prove (7), we start with the same inequalities as before, and then apply Lemma K twice, and Theorem 1. For $1 \leq p \leq 2$ we have

$$\begin{aligned} ||AXB - BXA + S||_p^p &= \left|\left| \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22} \end{bmatrix} \right|\right|_p^p \\ &\geq 2^{p-2}(||S_{11}||_p^p + ||S_{12}||_p^p + ||S_{21}||_p^p \\ &\quad + ||A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22}||_p^p) \\ &\geq 2^{p-2}(||S_{11}||_p^p + ||S_{12}||_p^p + ||S_{21}||_p^p + ||S_{22}||_p^p) \\ &\geq 2^{p-2}\, ||S||_p^p\,, \end{aligned}$$

and for $2 \leq p < \infty$ we have

$$\begin{aligned} ||AXB - BXA + S||_p^p &= \left|\left| \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22} \end{bmatrix} \right|\right|_p^p \\ &\geq ||S_{11}||_p^p + ||S_{12}||_p^p + ||S_{21}||_p^p \\ &\quad + ||A_2X_{22}B_2 - B_2X_{22}A_2 + S_{22}||_p^p \\ &\geq ||S_{11}||_p^p + ||S_{12}||_p^p + ||S_{21}||_p^p + ||S_{22}||_p^p \\ &\geq 2^{2-p}\, ||S||_p^p\,. \end{aligned}$$

Therefore $||AXB - BXA + S||_p^p \geq 2^{-|2-p|}\, ||S||_p^p$ which is equivalent to (7).

If $p = 2$, (7) becomes $||E(X) + S||_2 \geq ||S||_2$, and according to Remark 1 this implies (8).

The proof is complete.

Theorem 3. *Let A, B, C, $D \in B(H)$ be normal operators, such that $AC = CA$, $BD = DB$, and define $E : B(H) \to B(H)$ by $E(X) = AXB + CXD$. If $S \in \ker E$, then*

(*a*) *the inequality* (6) *holds in every u.i. norm;*
(*b*) *the inequality* (7) *is true, where* $||\quad||_p$ *is the* $\mathcal{C}_p$*-norm;*
(*c*) *the equality* (8) *is true;*
(*d*) *the inequality* (3) *holds in every u.i. norm, provided that*

$$A^*A + C^*C > 0 \quad \text{and} \quad B^*B + D^*D > 0. \tag{9}$$

Proof. It is enough to take the Hilbert space $H \oplus H$, and operators

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} -C & 0 \\ 0 & B \end{bmatrix}; \quad \tilde{S} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}; \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$$

and to apply Theorems 1 and 2.

4. Some corollaries

If $B = -C = I$, the condition (9) is satisfied and so, from Theorem 3, we obtain the well-known Theorem AK.

If $C = -D = I$, the conditions $AC = CA$, $BD = DB$ and (9) are satisfied, and from Theorem 3 we obtain the result:

Corollary 1. *If A, $B \in B(H)$ are normal, then for every operator S satisfying $ASB = S$ and all operator $X \in B(H)$ the inequality* (3) *holds.*

This is, in fact, the major result given in [8] by B. P. Duggal (Theorem 1 and Corollary 1). This is also a refinement of a result of H. Du [5] who proved under the same assumptions that

$$||AXB - X + S|| \geq ||A||^{-1} \, ||B||^{-1} \, ||S|| \,. \tag{10}$$

Indeed, if $||A|| \, ||B|| < 1$, from $ASB = S$ follows $||S|| = ||ASB|| \leq ||A|| \, ||B|| \, ||S||$ implying $S = 0$. Hence (10) is of interest only if $||A|| \, ||B|| \geq 1$, but then (3) is sharper than (10).

Corollary 2. *If the assumptions of Theorem* 3 *hold, then $\overline{\operatorname{ran} E} \cap \ker E = \{0\}$, where the closure can be taken in the most weak (uniform) norm. Hence $E(E(X)) = 0$ implies $E(X) = 0$.*

Proof. Indeed, if $Y \in \overline{\operatorname{ran} E} \cap \ker E$, then $Y = \lim_{n \to +\infty} E(X_n)$ and $E(Y) = 0$. From Theorem 3 we have $||E(X_n) - Y|| \geq c \, ||Y||$, and hence $||Y - Y|| \geq c \, ||Y||$. Thus $Y = 0$.

Remark 3. The smallest positive integer n such that $\ker E^n = \ker E^{n+1}$ is called the ascent of the operator E ($\operatorname{asc} E$). In view of this Corollary 2 asserts that $\operatorname{asc} E \leq 1$, under the assumptions of Theorem 3.

As it is easily seen from [6], for operators of the form $E(X) = \sum_{j=1}^{n} A_j X B_j$, where A_j and B_j are commuting families of normal operators, which satisfy $\operatorname{asc} E \leq 1$, the generalized Fuglede-Putnam theorem holds, i.e. $\sum_{j=1}^{n} A_j X B_j = 0$ implies $\sum_{j=1}^{n} A_j^* X B_j^* = 0$. So, we get as an easy consequence the following well-known result.

Corollary 3. *If A, B, C, D are normal operators such that $AC = CA$, $BD = DB$, then $AXB + CXD = 0$ implies $A^*XB^* + C^*XD^* = 0$.*

Results analogous to those given in Theorem 1 and Theorem 2 can be obtained if the normality of A and B is replaced by some other condition.

Corollary 4. *If $AB = BA$ and if the quadratic forms $\langle Ax, Bx \rangle$, $\langle A^*x, B^*x \rangle$ are real, then the relations* (6), (7) *and* (8) *hold. Moreover, if $\ker A \cap \ker B = \ker A^* \cap \ker B^* = \{0\}$, then the inequality* (3) *is valid.*

Proof. Indeed, it is enough to apply Theorem 1 and Theorem 2 to the operators $\tilde{A}$, $\tilde{B}$, $\tilde{X}$, $\tilde{S}$ defined on $H \oplus H$ by

$$\tilde{A} = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix}; \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}; \quad \tilde{S} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

Corollary 5. *If* $|A| = |B|$, $|A^*| = |B^*|$, *then the relations* (6), (7) *and* (8) *hold. Moreover, if* $\ker A \cap \ker B = \ker A^* \cap \ker B^* = \{0\}$, *then the inequality* (3) *is valid.*

Proof. Indeed, we apply Theorem 1 and Theorem 2 to the operators $\tilde{A}$, $\tilde{B}$, $\tilde{X}$, $\tilde{S}$ defined on $H \oplus H$ by

$$\tilde{A} = \begin{bmatrix} 0 & A^* \\ B & 0 \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} 0 & B^* \\ A & 0 \end{bmatrix}; \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}; \quad \tilde{S} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

5. A comment and some open questions

Operators of the form $E(X) = \sum_{j=1}^n A_j X B_j$ are known as 'elementary operators'. For these operators we may define the generalized adjoint by $E^*(X) = \sum_{j=1}^n A_j^* X B_j^*$, and we say that E is normal if $EE^* = E^*E$. (In the case when $E : \mathcal{C}_2 \to \mathcal{C}_2$, this definition coincides with the usual one.) It is easy to see that $TT^* = T^*T$ is a necessary and sufficient condition for the mapping δ_T, defined by $\delta_T(X) = TX - XT$, to be normal. Also, $AC = CA$, $BD = DB$ together with $AA^* = A^*A$, $BB^* = B^*B$, $CC^* = C^*C$, $DD^* = D^*D$ ensures that the mapping $E(X) = AXC + CXD$ is normal. In general, elementary operators $E(X) = \sum_{j=1}^n A_j X B_j$, where A_j and B_j are commuting families of normal operators, are called 'normally represented elementary operators', and it is easy to see that every normally represented elementary operator is normal.

The following questions naturally arise:

1. Is the condition that E is normally represented necessary and sufficient for its normality?
2. Direct generalization of Theorem 3 to the operators $E(X) = \sum_{j=1}^n A_j X B_j$ for $n > 2$ is not possible. In [7] Shulman stated that there exists a normally represented elementary operator of the form $\sum_{j=1}^n A_j X B_j$ with $n > 2$ such that $\operatorname{asc} E > 1$, i.e. the range and the kernel have nontrivial intersection. Nevertheless, for a normally represented operator $E(X) = \sum_{j=1}^n A_j X B_j$ with $n > 2$, we conjecture that the range of the operator E^{n-1} is orthogonal to the kernel of the operator E.
3. Anderson [1] showed that the equality $B(H) = \overline{\operatorname{ran} \delta_N} \oplus \ker \delta_N$ is true only in some very special cases, namely when the spectrum of N is finite. It would be interesting to find the condition which ensures the validity of $\mathcal{J} = \overline{\operatorname{ran} E} \oplus \ker E$, for the operator $E(X) = AXB + CXD$, considered as an operator mapping $\mathcal{J}$ into $\mathcal{J}$, where $\mathcal{J}$ is an ideal of compact operators.

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