PROJECTIVE BOUNDEDNESS AND CONVOLUTION
OF FRÉCHET MEASURES

R. BLEI AND J. CAGGIANO

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Abstract. Fréchet measures of order \( n \) (\( F_n \)-measures) are the measure-theoretic analogues of bounded \( n \)-linear forms on products of \( C_0(K) \) spaces. In an LCA setting, convolution of \( F_2 \)-measures is always defined, while there exist \( F_3 \)-measures whose convolution cannot be defined. In a three-dimensional setting, we demonstrate the existence of an \( F_2 \)-measure which cannot be convolved with arbitrary \( F_3 \)-measures.

Let \((X_1, A_1), \ldots, (X_n, A_n)\) be measurable spaces. A scalar-valued function \( \mu \) on \( A_1 \times \cdots \times A_n \) is a Fréchet measure (an \( F_n \)-measure) if, when any \( n-1 \) coordinates are fixed, \( \mu \) is a measure in the remaining coordinate. When the measure spaces are arbitrary or understood, we write \( F_n \) for \( F_n(A_1, \ldots, A_n) \).

Let \( X_1; \ldots, X_n \) be locally compact abelian groups with corresponding dual groups \( \hat{X}_1; \ldots, \hat{X}_n \). If \( A_1; \ldots, A_n \) are the respective Borel fields of \( X_1; \ldots, X_n \) and \( \mu \in F_n \), then the Fourier-Stieltjes transform of \( \mu \) is given by

\[
\hat{\mu}(\gamma_1, \ldots, \gamma_n) = \int \gamma_1 \otimes \cdots \otimes \gamma_n d\mu, \quad (\gamma_1, \ldots, \gamma_n) \in \hat{X}_1 \times \cdots \times \hat{X}_n.
\]

(The integral above is defined iteratively.) We use the multi-linear Riesz Representation Theorem \( [B1] \) to identify \( F_n(A_1, \ldots, A_n) \) with the dual space of

\[
C_0(X_1) \otimes \cdots \otimes C_0(X_n) = V_n(X_1, \ldots, X_n),
\]

and thus extend the action to arbitrary bounded functions by integration. See \( [B3] \) for details. It is natural to consider the feasibility of convolution of \( F_n \)-measures.

Definition 1. Let \( X_1; \ldots, X_n \) be LCA groups with Borel fields \( A_1; \ldots, A_n \). \( F_n \)-measures \( \mu \) and \( \nu \) are **convolvable** if \( \hat{\mu} \hat{\nu} = \lambda \) for some \( \lambda \in F_n \); \( \lambda \) is then denoted by \( \mu \ast \nu \) or \( \nu \ast \mu \). We say \( \mu \in F_n \) is a **convolver** if \( \mu \ast \nu \) exists for all \( \nu \in F_n \).

The case \( n = 1 \) in Definition 1 is classical; every \( F_1 \)-measure is a convolver. It is shown in \( [GS1] \) that in a two-dimensional setting every \( F_2 \)-measure is a convolver, while in \( [GS2] \) it is shown that there exist non-convolvers in \( F_3 \). In general, convolvability is related to **projective boundedness**, a property conveying a Grothendieck-type inequality. In the definition which follows, \( \mathcal{L}^\infty(X_j) \) denotes the space of bounded functions on \( X_j \).

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Definition 2 \([\text{B3}]\). Let \((\mathcal{X}_1, \mathcal{A}_1), \ldots, (\mathcal{X}_n, \mathcal{A}_n)\) be measurable spaces, and let \(\mu \in \mathcal{F}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)\). For \(f_j \in L^\infty(\mathcal{X}_j)\), \(j = 1, \ldots, n\), define
\[
\phi_\mu(f_1, \ldots, f_n) = \int f_1 \otimes \cdots \otimes f_n d\mu.
\]
We say that \(\mu\) is projectively bounded if
\[
\|\mu\|_{pb} := \sup\{\|\phi_\mu\|_{\mathcal{V}_n(\mathcal{E}_1, \ldots, \mathcal{E}_n)} : E_j \subset Ball(L^\infty(\mathcal{X}_j)), |E_j| < \infty, j = 1, \ldots, n\} < \infty;
\]
the class of projectively bounded \(\mathcal{F}_n\)-measures on \(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n\) is denoted by \(\mathcal{PBF}_n = \mathcal{PBF}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)\).

Let \(\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_n\) be locally compact Hausdorff spaces with respective Borel fields \(\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}_1, \ldots, \mathcal{B}_n\). For
\[
f \in \mathcal{V}_n(\mathcal{X}_1 \times \mathcal{Y}_1, \ldots, \mathcal{X}_n \times \mathcal{Y}_n) \quad \text{and} \quad \mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n),
\]
define a function \(\eta_{f, \mu}\) on \(\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n\) by
\[
\eta_{f, \mu}(y_1, \ldots, y_n) = \int f(x_1, y_1, \ldots, x_n, y_n) d\mu(d x_1, \ldots, d x_n) \quad (y_j \in \mathcal{Y}_j).
\]

Definition 3. Let \(\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)\).

1. (\(\tau\)-projective boundedness) Let \(\mathcal{Y}_1, \ldots, \mathcal{Y}_n\) be locally compact Hausdorff spaces, and define
\[
\|\mu\|_{pb_{\tau,n}} := \sup\{\|\eta_{f, \mu}\|_{\mathcal{V}_n(\mathcal{Y}_1, \ldots, \mathcal{Y}_n)} : \|f\|_{\mathcal{V}_n(\mathcal{X}_1 \times \mathcal{Y}_1, \ldots, \mathcal{X}_n \times \mathcal{Y}_n)} \leq 1\}.
\]
We say that \(\mu\) is \(\tau\)-projectively bounded if \(\|\mu\|_{pb_{\tau,n}} < \infty\); the class of \(\tau\)-projectively bounded \(\mathcal{F}_n\)-measures on \(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n\) is denoted by \(\mathcal{PBF}_\tau,n = \mathcal{PBF}_\tau,n(\mathcal{A}_1, \ldots, \mathcal{A}_n)\).

2. (\(g\)-projective boundedness) Let \(\mathcal{X}_1, \ldots, \mathcal{X}_n\) be LCA groups. For \(\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)\) and elementary tensors \(f = f_1 \otimes \cdots \otimes f_n \in \mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)\), let
\[
\Psi_{f, \mu}(y_1, \ldots, y_n) = \int f(x_1 + y_1, \ldots, x_n + y_n) d\mu(d x_1, \ldots, d x_n) \quad (y_j \in \mathcal{X}_j),
\]
and define
\[
\|\mu\|_{pb_{g,n}} := \sup\{\|\Psi_{f, \mu}\|_{\mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)} : \|f\|_{\mathcal{V}_n} = \prod_j \|f_j\|_\infty \leq 1\}.
\]
We say that \(\mu\) is \(g\)-projectively bounded if \(\|\mu\|_{pb_{g,n}} < \infty\); the class of \(g\)-projectively bounded \(\mathcal{F}_n\)-measures on \(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n\) is denoted by \(\mathcal{PBF}_{g,n} = \mathcal{PBF}_{g,n}(\mathcal{A}_1, \ldots, \mathcal{A}_n)\).

The symbols \(\tau\) and \(g\) in Definition 3 denote, respectively, topological and group-topological projective boundedness. It is straightforward to check that \(\mathcal{PBF}_n, \mathcal{PBF}_{g,n}\), and \(\mathcal{PBF}_{\tau,n}\) are Banach spaces. To verify that every \(\mu \in \mathcal{PBF}_{g,n}\) is a convolver, let \(\nu \in \mathcal{F}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)\) be arbitrary and consider a linear form \(\Lambda : \mathcal{V}_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) \to \mathbb{C}\) given by
\[
\Lambda(f) = \int \left(\int f(x_1 + y_1, \ldots, x_n + y_n) d\mu(d x_1, \ldots, d x_n)\right) \nu(dy_1, \ldots, dy_n).
\]
Then $\Lambda$ is bounded, and the $\mathcal{F}_n$-measure $\lambda$ representing this linear action satisfies $\lambda = \mu \nu$. Each $\mu \in \mathcal{PBF}_{g,n}$ can be identified with a bounded linear operator $T_\mu : \mathcal{F}_n \to \mathcal{F}_n$ given by $T_\mu(\nu) = \mu \ast \nu$. By a routine argument, $\mu \in \mathcal{PBF}_{g,n}$ if and only if the domain of $T_\mu$ is $\mathcal{F}_n$, and thus $\mathcal{PBF}_{g,n}$ is precisely the space of convolvers.

For $\mu \in \mathcal{F}_n$, let $\mathcal{D}(\mu) = \{ \alpha \in \mathcal{F}_n : \mu \ast \alpha \text{ exists} \}$, i.e., $\mathcal{D}(\mu)$ is the domain of $T_\mu$. Results in [GS1] and [GS2] state that in a two-dimensional setting $\mathcal{D}(\mu) = \mathcal{F}_2$ for all $\mu \in \mathcal{F}_2$, and there exist $\mu \in \mathcal{F}_3$ with $\mathcal{D}(\mu) \subsetneq \mathcal{F}_3$. Thus it is natural to ask: for $\mu \in \mathcal{F}_2(\sigma(A_1 \times A_2), A_3)$, is $\mathcal{D}(\mu) = \mathcal{F}_3(A_1, A_2, A_3)$? We show that the answer is no.

Let $\mathcal{S}(A_j)$ be the space of $A_j$-simple functions on $X_j$, equipped with the uniform norm. $\mathcal{V}_n(A_1, \ldots, A_n)$ will denote the completion of $\mathcal{S}(A_1) \otimes \cdots \otimes \mathcal{S}(A_n)$ with respect to the projective tensor norm.

**Proposition 4.** $\mathcal{PBF}_n \subset \mathcal{PBF}_{\tau,n} \subset \mathcal{PBF}_{g,n}$.

**Proof.** The second inclusion is immediate from the definitions. Let $X_1, \ldots, X_n$ be compact spaces. A theorem of Saeki [S] states that

$$C(X_1 \times \cdots \times X_n) \cap \mathcal{V}_n(A_1, \ldots, A_n) = \mathcal{V}_n(X_1, \ldots, X_n).$$

We show that

$$\|\eta_{f,\mu}\|_{\mathcal{V}_n(Y_1, \ldots, Y_n)} \leq \|f\|_{\mathcal{V}_n} \|\mu\|_{\mathcal{PBF}_n}, \quad f \in \mathcal{V}_n(X_1 \times Y_1, \ldots, X_n \times Y_n).$$

(See (1) for the definition of $\eta_{f,\mu}$.) Let $g = g_1 \otimes \cdots \otimes g_n$ be an elementary tensor in $C_c(X_1 \times Y_1, \ldots, X_n \times Y_n)$. (The subscript $c$ denotes compact support.) Then $\eta_{g,\mu} \in C_c(Y_1 \times \cdots \times Y_n)$. For locally compact spaces $X$ and $Y$, $\mathcal{V}_n(X, Y)$ is uniformly dense in $C_0(X \times Y)$, and thus for $k = 1, \ldots, n$ there exist sequences $\{\rho_j\} \subset \mathcal{S}(A_k) \otimes \mathcal{S}(B_k)$ with $\lim_{j \to \infty} \rho_j = g_k$ in the uniform norm. Let $\theta_j = \rho_{j1} \otimes \cdots \otimes \rho_{jn}$. Then $\lim_j \eta_{\theta_j,\mu} = \eta_{g,\mu}$ (uniform limit), and $\{\eta_{\theta_j,\mu}\}$ is a $\mathcal{V}_n(B_1, \ldots, B_n)$-Cauchy sequence satisfying

$$\|\eta_{\theta_j,\mu}\|_{\mathcal{V}_n(B_1, \ldots, B_n)} \leq \|\rho_{j1}\| \cdots \|\rho_{jn}\| \|\mu\|_{\mathcal{PBF}_n}.$$ 

By (1), $\eta_{g,\mu} \in \mathcal{V}_n(Y_1, \ldots, Y_n)$. Let $\epsilon > 0$, and choose $f \in \mathcal{V}_n(X_1 \times Y_1, \ldots, X_n \times Y_n)$. Without loss of generality, we can take $f$ to be compactly supported. We represent

$$f = \sum_k g_k,$$

where $g_k = g_{1k} \otimes \cdots \otimes g_{nk}$, and

$$\|f\|_{\mathcal{V}_n} \leq (1 + \epsilon) \sum_k \|g_{1k}\| \cdots \|g_{nk}\|,$$

with each $g_k \in C_c(X_1 \times Y_1) \otimes \cdots \otimes C_c(X_n \times Y_n)$. Then $\eta_{f,\mu} = \sum_k \eta_{g_k,\mu}$, and (2) follows.

We do not know which of the inclusions in Proposition 4 are proper. ($\mathcal{PBF}_n$ is properly contained in $\mathcal{PBF}_{g,n}$, as shown by the ‘fractional’ forms in [B3].)

We now show that in a three-dimensional setting, an $\mathcal{F}_2$-measure need not be a convolver.

**Lemma 5.** Let $X_1$, $X_2$, and $X_3$ be infinite locally compact abelian groups with respective Borel fields $A_1, A_2, A_3$. For every $K > 0$ there exists a discrete measure $\mu$ on $X_1 \times X_2 \times X_3$ satisfying $\|\mu\|_{\mathcal{F}_2(A_1 \times A_2, A_3)} \leq 1$ and $\|\mu\|_{\mathcal{PBF}_{g,3}(A_1, A_2, A_3)} \geq K$. 
Proof. Let $\mathbb{Z}_n$ be the group of integers under addition modulo $n$, and let $\hat{\mathbb{Z}}_n$ be its dual. Let $\mu_n : C(\mathbb{Z}_n \times \mathbb{Z}_n) \to \mathbb{C}$ be given by

$$\mu_n(f, h) = \sum_{m \in \mathbb{Z}_n} \frac{f(m, m)}{\sqrt{n}} h(m).$$

We show that $\|\mu_n\|_{\mathcal{F}_2(\hat{\mathbb{Z}}_n \times \hat{\mathbb{Z}}_n, \mathbb{Z}_n)} \leq 1$ and $\|\hat{\mu}_n\|_{\mathcal{V}_3(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{Z}_n)} \to \infty$ as $n \to \infty$. (For convenience, we write $\mathcal{F}_2(\hat{\mathbb{Z}}_n \times \hat{\mathbb{Z}}_n)$ for $\mathcal{F}_2(\hat{\mathbb{Z}}_n \times \hat{\mathbb{Z}}_n, 2\mathbb{Z}_n)$.) Clearly, $\|\mu_n\|_{\mathcal{F}_2} \leq 1$ (by Cauchy-Schwarz). $\mu_n$ is also an element of $\mathcal{F}_3(\hat{\mathbb{Z}}_n, \mathbb{Z}_n, \mathbb{Z}_n)$:

$$\mu_n(f, g, h) = \sum_{m \in \mathbb{Z}_n} \frac{f(m)g(m)}{\sqrt{n}} h(m).$$

Then

$$\hat{\mu}_n(j, k, l) = \sum_{m \in \mathbb{Z}_n} e^{-\frac{2\pi i m j}{n}} e^{-\frac{2\pi i m k}{n}} \sum_{s \in \mathbb{Z}_n} e^{-\frac{2\pi i (j+m) s}{n}} = \frac{1}{\sqrt{n}} e^{2\pi ij/n} e^{2\pi ik/n}.$$ 

Define

$$\Phi_n(j, k, l) = \frac{1}{n^2} e^{-2\pi ij/n} e^{-2\pi ik/n}.$$ 

Let $s, t, u$ be elements in the unit ball of $\ell^\infty(\mathbb{Z}_n)$. Then

$$|\sum_{j, k, l} \Phi_n(j, k, l)s(j)t(k)u(l)|$$

$$= \frac{1}{n^2} \sum_l u(l) \sum_j s(j) e^{-2\pi ij/n} \sum_k t(k) e^{-2\pi ik/n}$$

$$= |\sum_l u(l)\hat{s}(l)\hat{t}(l)| \leq \|u\|_\infty \|\hat{s}\|_2 \|\hat{t}\|_2 \leq \|s\|_\infty \|t\|_\infty \leq 1.$$ 

Thus, $\|\Phi_n\|_{\mathcal{F}_3} \leq 1$. However,

$$|\sum_{j, k, l} \hat{\mu}_n(j, k, l)\Phi_n(j, k, l)| = \sqrt{n}.$$ 

Therefore, by the duality $(\mathcal{V}_3)^* = \mathcal{F}_3$,

$$\sqrt{n} \leq \|\hat{\mu}_n\|_{\mathcal{V}_3} \leq \|\mu_n\|_{\mathcal{F}_3}.$$ 

Let $[N] = \{1, \ldots, N\}$. Given $K > 0$, there exist $N$ and $\mu \in \mathcal{F}_2([N]^2, [N])$ with $\|\mu\|_{\mathcal{F}_2} \leq 1$ and $\|\mu\|_{\mathcal{F}_3} > K$. If $\mu = \{\mu_{xyz} : (x, y, z) \in [N]^3\}$, there exist arrays $a, b, c$ in the unit ball of $\ell^\infty([N]^3)$ such that

$$\|\eta_{a, b, c, \mu}\|_{\mathcal{V}_3([N], [N], [N])} > K,$$

where

$$\eta_{a, b, c, \mu}(i, j, k) = \sum_{(x,y,z)\in[N]^3} \mu_{xyz} a_x b_{yz} c_z, \quad (i, j, k) \in [N]^3.$$
Let $F_m$ and $G_m$ be disjoint mutually independent subsets of $X_m$ of cardinality $N$ given by

$$F_m = \{s_{jm} : j \in [N]\}, \quad G_m = \{t_{jm} : j \in [N]\}, \quad m = 1,2,3.$$  

(Disjoint subsets $A$ and $B$ of an abelian group are mutually independent if given elements $(a_1,b_1)$ and $(a_2,b_2)$ of $A \times B$, the relation $a_1 + b_1 = a_2 + b_2$ implies that $a_1 = a_2$ and $b_1 = b_2$.) Define a measure $\tilde{\mu}$ on $X_1 \times X_2 \times X_3$ by

$$\tilde{\mu} = \sum_{(x,y,z) \in [N]^3} \mu_{xyz} \delta_{s_{x1}} \otimes \delta_{s_{y2}} \otimes \delta_{s_{z3}},$$

and observe that $\|\tilde{\mu}\|_{\mathcal{F}_2(\sigma(A_1 \times A_2),A_3)} \leq 1$. Therefore, by the independence of $F_m$ and $G_m$, we can find $f \in C_0(X_1)$, $g \in C_0(X_2)$, and $h \in C_0(X_3)$ with

$$f(s_{i1} + t_{j1}) = a_{ij}, \quad g(s_{i2} + t_{j2}) = b_{ij}, \quad h(s_{i3} + t_{j3}) = c_{ij},$$

so that

$$\eta_{a,b,c;\mu}(i,j,k) = \Psi_{f \otimes g \otimes h;\mu}(t_{i1},t_{j2},t_{k3}), \quad (i,j,k) \in [N]^3.$$  

(Refer to \cite{B3} and \cite{B2} for definitions of $\eta$ and $\Psi$.) Thus $\|\Psi_{f \otimes g \otimes h;\mu}\|_{\mathcal{V}_3} \geq K$, and hence $\|\tilde{\mu}\|_{\mathcal{P}_{b,\nu}} \geq K$. \hfill \Box

**Corollary 6.** If the underlying $\sigma$-algebras $A_1, A_2,$ and $A_3$ are infinite, there exists $\mu \in \mathcal{F}_2(\sigma(A_1 \times A_2),A_3)$ which is not a convolver in $\mathcal{F}_3$.

**Two further questions**

1. It is shown in \cite{B3} that scalar measures are projectively bounded. Hence, by Proposition 2 all scalar measures are convolvers. An $\mathcal{F}_3$-measure $\mu$ is an $\mathcal{F}_2$-measure if, when any one coordinate is fixed, $\mu$ extends to a scalar measure in the remaining two coordinates. (See \cite{B2} for details.) We have the proper containments $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$. We do not know whether all $\mathcal{F}_2$-measures are convolvers.

2. Let $X_1, \ldots, X_n$ be LCA groups. It is shown in \cite{ZS} that the space of completely bounded $n$-linear forms on $C_0(X_1) \times \cdots \times C_0(X_n)$ has a natural Banach *-algebra structure extending that of $\mathcal{F}_1$ on $X_1 \times \cdots \times X_n$. We do not know whether all completely bounded forms are convolvers, or if all convolvers are completely bounded.

**References**


Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269
E-mail address: Blei@uconnvm.uconn.edu

Department of Mathematics & Computer Science, Arkansas State University, Box 70, State University, Arkansas 72467
E-mail address: Caggiano@csu.astate.edu