SOME NEW EMBEDDINGS AND NONIMMERSIONS
OF REAL PROJECTIVE SPACES

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(Communicated by Ralph Cohen)

ABSTRACT. We obtain new families of embeddings and nonimmersions of real projective spaces in Euclidean space. The method involves obstruction theory, and includes several new insights.

1. STATEMENT OF RESULTS

In this paper, we obtain the following new results regarding immersions and embeddings of real projective space P^m in Euclidean space. Let $\alpha(n)$ denote the number of 1's in the binary expansion of n.

Theorem 1.1. *If $\alpha(n) = 2$, then*

(1) P^{16n+8} *cannot be immersed in* $\mathbf{R}^{32n+3}$*, and*
(2) P^{16n+10} *cannot be immersed in* $\mathbf{R}^{32n+11}$*.*

Theorem 1.2. *If $\alpha(n) > 2$, then P^{8n+4} can be embedded in $\mathbf{R}^{16n+1}$.*

Theorem 1.1(1) improves on the previously best known result ([1]) by 1 dimension, while Theorem 1.1(2) improves on the previously best known nonimmersion and nonembedding results ([2]) for P^{16n+10} and P^{16n+11} by 4 dimensions, and is within 1 of best possible for them. It also implies new nonimmersions for P^{16n+12}, P^{16n+13}, and P^{16n+14}. Theorem 1.2 improves on the previously best known embedding ([11]) of P^{8n+4} by 1 dimension. These results can best be appreciated when viewed in a table of known embedding and immersion results for real projective spaces. Such a table may be seen on the internet (see [3]).

The method of proof is obstruction theory, specifically modified Postnikov towers (MPTs). The reason that Theorem 1.1 had not been noticed before is that the first author used to think that nonimmersions were extremely difficult to prove by MPTs because of the possibility of secondary and higher order indeterminacy. We show here that sometimes there is no secondary or higher order indeterminacy for a simple reason. The reason that Theorem 1.2 had not been noticed before is that this was apparently the first time that Mahowald's inductive approach to constructing embeddings ([8]) was combined with the method of evaluation of obstructions initiated in [5].

Theorem 1.1(1) and Theorem 1.2 form the bulk of the second author's thesis ([12]), written under the direction of the first author, who subsequently discovered the second part of Theorem 1.1.

2. Proof of nonimmersions

In this section, we prove Theorem 1.1(2). The proof of Theorem 1.1(1), which was detailed in [12], is extremely similar, and is omitted.

The problem is reduced to obstruction theory by the following result of Sanderson, which reduces the immersion question to the determination of the geometric dimension of a multiple of the Hopf bundle ξ_n over P^n.

Proposition 2.1 ([10]). *P^n can be immersed in $\mathbf{R}^{n+k}$ if and only if the map $P^n \to BO$ which classifies $(n + k + 1)\xi_n$ can be factored as $P^n \to BO(k) \to BO$.*

Thus Theorem 1.1(2) will follow from the following result, whose proof will occupy the rest of this section.

Theorem 2.2. *If $\alpha(n) = 2$, then $(32n + 12)\xi : P^{16n+10} \to BO$ does not factor through $BO(16n + 1)$.*

The map of Theorem 2.2 factors as

$$(2.3) \qquad P^{16n+10} \to HP^{4n+2} \overset{(8n+3)H}{\to} BSp \to BO.$$

Here HP^m denotes quaternionic projective space, and pH a multiple of the quaternionic Hopf bundle. We let $\widetilde{BSp}(n)$ denote the classifying space for quaternionic bundles of real geometric dimension n; it is the pullback of maps of $BO(n)$ and BSp to BO. We will prove Theorem 2.2 by showing that the map $P^{16n+10} \to BSp$ in (2.3) does not lift to $\widetilde{BSp}(16n+1)$. This will be accomplished using the following MPT. For typographical reasons, we abbreviate $K(\mathbf{Z}_2, 16n+i)$ as K_i. We will, however, name the corresponding k-invariant $k_{16n+i}^j \in H^{16n+i}(E_j)$. (All coefficients are in $\mathbf{Z}/2$.) All of our MPTs are performed through the range of dimensions relevant for the real projective space being mapped in, here $16n + 10$. This will always be well within the stable range.

$$\widetilde{BSp}(16n + 1)$$
$$\downarrow$$
$$E_3$$
$$\downarrow$$
$$K_2 \times K_3 \times K_4 \times K_6 \times K_7 \times K_8 \times K_8' \searrow E_2 \to K_4 \times K_8 \times K_8' \times K_9 \times K_{10} \times K_{10}'$$
$$\downarrow$$
$$K_1 \times K_3 \times K_7 \searrow E_1 \to K_3 \times K_4 \times K_5 \times K_7 \times K_8 \times K_9 \times K_9'$$
$$\downarrow$$
$$P^{16n+10} \to HP^{4n+2} \overset{(8n+3)H}{\to} BSp \to K_2 \times K_4 \times K_8$$
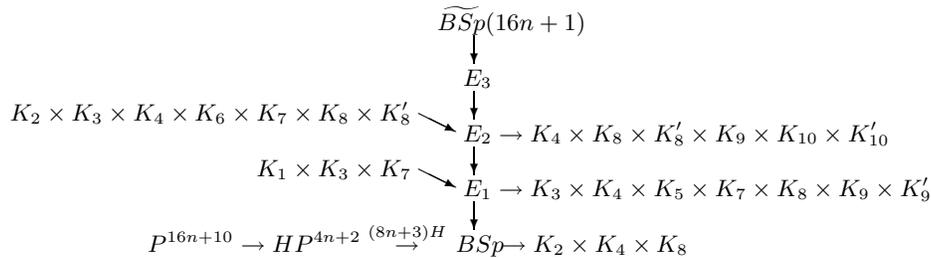
FIGURE 2.4.

MPTs were introduced in [8] and [6]. Each vertical map in the above diagram is part of a fiber sequence preceded by the map from the fiber represented by a diagonal arrow, and followed by the classifying map represented by a horizontal arrow. The information of the diagram can be obtained from the ASS of the stunted real projective space P_{16n+1}, which is, in the stable range, the fiber of

$\widetilde{BSp}(16n + 1) \to BSp$. This ASS can be found in Table 8.2 on page 54 of [9]. We re-create it in Figure 2.5.
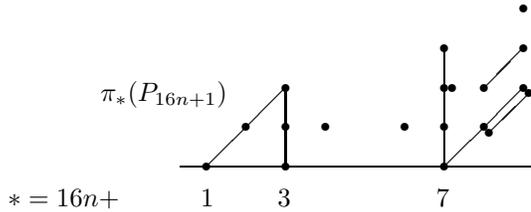

FIGURE 2.5.

The generalized Eilenberg-MacLane spaces (GEMs) on the right side of Figure 2.4 give rise to the k-invariants for the MPT; these classes have dimensions 1 greater than those of the corresponding elements of $\pi_*(P_{16n+1})$. We will prove that the map $P^{16n+10} \to BSp$ lifts to E_2, and every lifting to E_2 sends some of the level-2 k-invariants nontrivially. Hence the map does not lift to E_3 or to $\widetilde{BSp}(16n + 1)$.

The method of evaluation of obstructions is to use the following result of [5]. Here bo is the spectrum for connective ko-theory localized at 2.

Theorem 2.6 ([5, 1.8]). *Let $B^o(m)$ denote the fiberwise smash product of $\widetilde{BSp}(m)$ with bo. In the stable range, there is a map of fibrations which is the natural inclusion on the fibers*

$$
\begin{array}{ccc}
P_m & \longrightarrow & P_m \wedge bo \\
\downarrow & & \downarrow \\
\widetilde{BSp}(m) & \longrightarrow & B^o(m) \\
\downarrow & & \downarrow \\
BSp & \overset{=}{\longrightarrow} & BSp
\end{array}
$$

If pH_k denotes p times the Hopf bundle over HP^k, then pH_k lifts to $B^o(m)$ if and only if $m \geq 2k$ and for all $i \leq k$

$$\nu\left(\binom{p}{i}\right) \geq \nu(|ko_{4i-1}(P_m)|).$$

Here, and throughout, $\nu(-)$ denotes the exponent of 2.

We use Theorem 2.6 to prove the following result.

Theorem 2.7. *In the MPT of Figure 2.4, there is a lifting of $(8n + 3)H_{4n+2}$ to E_2 sending k^2_{16n+4} and k^2_{16n+8} nontrivially.*

Proof. We begin by observing that $\nu\binom{8n+3}{4n+1} = \nu\binom{8n+3}{4n+2} = \alpha(n) = 2$. Indeed, both are equal to $\alpha(4n + 1) + \alpha(4n + 2) - \alpha(8n + 3)$.

These must be compared with the numbers in the following table. Results such as these have been tabulated in many papers of the first author, such as [5].

$\nu(ko_{4i-1}(P_m))$	m			
	$16n+1$	$16n+2$	$16n+5$	$16n+6$
$i = 4n+1$	3	2	0	0
$i = 4n+2$	4	4	3	2

The orders tabulated here correspond to ASS charts pictured below, which may be computed, for example, as in [4].

$* = 16n+$

$ko_*(P_{16n+1})$ $ko_*(P_{16n+2})$ $ko_*(P_{16n+5})$ $ko_*(P_{16n+6})$
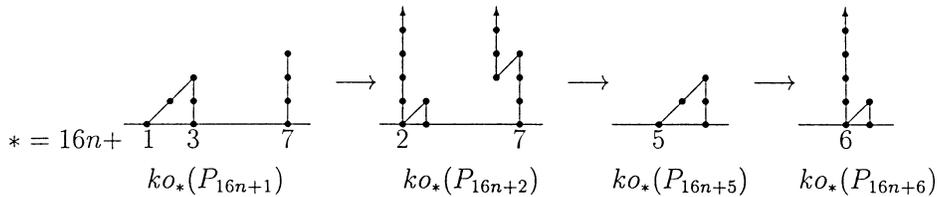
FIGURE 2.8.

There are MPTs for the fibrations $B^o(16n+\delta) \to BSp$, and induced maps of the MPTs, whose spaces we denote by $E_s^o(16n+\delta)$. By Theorem 2.6, $(8n+3)H_{4n+1}$ lifts to $B^o(16n+2)$ but not to $B^o(16n+1)$. This implies that $(8n+3)H_{4n+1}$ lifts to $E_2^o(16n+1)$ sending k_{16n+4}^2 nontrivially. (Keep in mind that degrees of k-invariants are 1 greater than those of corresponding elements of π_*(fiber), which in this case is $ko_*(P_{16n+\delta})$.) Similarly, $(8n+3)H_{4n+2}$ lifts to $B^o(16n+6)$, but not to $B^o(16n+5)$. All this implies that $(8n+3)H_{4n+2}$ lifts to $E_2^o(16n+1)$, sending both k_{16n+4}^2 and k_{16n+8}^2 nontrivially.

The map $\widetilde{BSp}(16n+1) \to B^o(16n+1)$ induces a map of MPTs. The mapping of k-invariants corresponds to the map of fibers $\pi_*(P_{16n+1}) \to ko_*(P_{16n+1})$, which are pictured in Figures 2.5 and 2.8. Various methods of computing Ext homomorphisms can be used to show this morphism is surjective in the indicated range. If F denotes the fiber of $E_2 \to E_2^o(16n+1)$, then $\pi_*(F)$ corresponds to elements in filtration 0 and 1 in Figure 2.5 which map trivially to $ko_*(P_{16n+1})$. Such elements occur only in filtration 1 and in homotopy dimension $16n+4$, $16n+6$, and $16n+8$. The obstructions to pulling the map $HP^{4n+2} \to E_2^o(16n+1)$ back to E_2 occur in $H^*(HP^{4n+2}; \pi_{*-1}(F))$, which is 0 since $\pi_{*-1}(F) = 0$ when $* \equiv 0 \mod 4$ and $* \leq 16n+8$. Thus $(8n+3)H_{4n+2}$ lifts to E_2. The k-invariants k_{16n+4}^2 and k_{16n+8}^2 in $H^*(E_2)$ are the images of corresponding k-invariants in $H^*(E_2^o(16n+1))$, which have already been shown to map nontrivially to $H^*(HP^{4n+2})$. Thus k_{16n+4}^2 and k_{16n+8}^2 map nontrivially, as claimed. We make no claim about whether or not $k_{16n+8}^{2\prime}$, which corresponds to the split $\mathbf{Z}/2$ in $\pi_{16n+7}(P_{16n+1})$ in filtration 2, maps nontrivially, since it is not in the image from $H^*(E_2^o(16n+1))$. □

In order to determine the indeterminacy for lifting P^{16n+10} in this MPT, we must know the relations which give rise to the k-invariants. These are computed by the method initiated in [6] and utilized in many subsequent papers by the first author and also in papers of Lam and/or Randall. It is a matter of building a minimal resolution using Massey-Peterson algebras. The relations for the MPT in Figure 2.4 are given in the table below.

w_{16n+2}	
w_{16n+4}	
w_{16n+8}	
$k^1_{16n+3}:$	$\mathrm{Sq}^2\, w_{16n+2}$
$k^1_{16n+4}:$	$\mathrm{Sq}^1\, w_{16n+4} + \mathrm{Sq}^2\, \mathrm{Sq}^1\, w_{16n+2}$
$k^1_{16n+5}:$	$(\mathrm{Sq}^4 + w_4)w_{16n+2}$
$k^1_{16n+7}:$	$(\mathrm{Sq}^4 + w_4)w_{16n+4}$
$k^1_{16n+8}:$	$\mathrm{Sq}^1\, w_{16n+8} + \mathrm{Sq}^2\, \mathrm{Sq}^3\, w_{16n+4}$
$k^1_{16n+9}:$	$\mathrm{Sq}^2\, w_{16n+8} + (\mathrm{Sq}^4 + w_4)\, \mathrm{Sq}^2\, w_{16n+4}$
$k^{1'}_{16n+9}:$	$(\mathrm{Sq}^8 + w_8)w_{16n+2} + w_4\, \mathrm{Sq}^2\, w_{16n+4}$
$k^2_{16n+4}:$	$\mathrm{Sq}^2\, k^1_{16n+3} + \mathrm{Sq}^1\, k^1_{16n+4}$
$k^2_{16n+8}:$	$\mathrm{Sq}^1\, k^1_{16n+8} + \mathrm{Sq}^2\, \mathrm{Sq}^3\, k^1_{16n+4}$
$k^{2'}_{16n+8}:$	$(\mathrm{Sq}^4 + \mathrm{Sq}^3\, \mathrm{Sq}^1 + w_4)k^1_{16n+5} + (\mathrm{Sq}^6 + w_4\, \mathrm{Sq}^2)k^1_{16n+3}$
$k^2_{16n+9}:$	$\mathrm{Sq}^2\, \mathrm{Sq}^1\, k^1_{16+7} + (\mathrm{Sq}^4 + w_4)\, \mathrm{Sq}^1\, k^1_{16n+5} + (\mathrm{Sq}^6 + w_4\, \mathrm{Sq}^2)k^1_{16n+4}$
	$+ (\mathrm{Sq}^4\, w_4)\, \mathrm{Sq}^2\, \mathrm{Sq}^1\, k^1_{16n+3}$
$k^2_{16n+10}:$	$\mathrm{Sq}^2\, k^1_{16n+9} + \mathrm{Sq}^3\, k^1_{16n+8} + (\mathrm{Sq}^4 + \mathrm{Sq}^3\, \mathrm{Sq}^1 + w_4)k^1_{16n+7}$
$k^{2'}_{16n+10}:$	$\mathrm{Sq}^2\, k^{1'}_{16n+9} + (\mathrm{Sq}^5\, \mathrm{Sq}^1 + \mathrm{Sq}^4\, \mathrm{Sq}^2)k^1_{16n+5} + w_4\, \mathrm{Sq}^3\, k^1_{16n+4}$
	$+ (\mathrm{Sq}^8 + w_8 + w_4\, \mathrm{Sq}^4 + w_4^2 + w_4\, \mathrm{Sq}^3\, \mathrm{Sq}^1)k^1_{16n+3}$
$k^3_{16n+8}:$	$\mathrm{Sq}^1\, k^2_{16n+8} + \mathrm{Sq}^2\, \mathrm{Sq}^3\, k^2_{16n+4}$
$k^3_{16n+10}:$	$\mathrm{Sq}^2\, k^2_{16n+9} + \mathrm{Sq}^2\, \mathrm{Sq}^1\, k^{2'}_{16n+8}$
	$+ (\mathrm{Sq}^7 + \mathrm{Sq}^4\, \mathrm{Sq}^2\, \mathrm{Sq}^1 + w_4(\mathrm{Sq}^3 + \mathrm{Sq}^2\, \mathrm{Sq}^1))k^2_{16n+4}$
$k^4_{16n+10}:$	$\mathrm{Sq}^2\, \mathrm{Sq}^1\, k^3_{16n+8}$

The lifting $f_2 : P^{16n+10} \to E_2$ can be varied through the fiber F_1 of $E_2 \to E_1$, which is the GEM in Figure 2.4 which ends with K'_8. This primary indeterminacy is computed using the above relations. For the bundle $(32n + 12)\xi$, both w_4 and w_8 are nonzero. Varying through K_{16n+2} changes $f_2^*(k^2_{16n+4})$ and $f_2^*(k^{2'}_{16n+8})$ and no other level-2 k-invariants. We illustrate why this is true in the second case.

The relation for $k^{2'}_{16n+8}$ means that the action map $\mu : F_1 \times E_2 \to E_2$ sends $k^{2'}_{16n+8}$ to

$$1 \otimes k^{2'}_{16n+8} + (\mathrm{Sq}^4 + \mathrm{Sq}^3\, \mathrm{Sq}^1)\iota_{16n+4} \otimes 1 + \iota_{16n+4} \otimes w_4 + \mathrm{Sq}^6\, \iota_{16n+2}$$
$$\otimes 1 + \mathrm{Sq}^2\, \iota_{16n+2} \otimes w_4.$$

The composite

$$(2.9) \qquad P^{16n+10} \overset{x^{16n+2} \times f_2}{\to} F_1 \times E_2 \overset{\mu}{\to} E_2$$

sends $k^{2'}_{16n+8}$ to

$$f_2^*(k^{2'}_{16n+8}) + \mathrm{Sq}^6(x^{16n+2}) + \mathrm{Sq}^2(x^{16n+2}) \cdot x^4 = f_2^*(k^{2'}_{16n+8}) + 0 + x^{16n+8}.$$

Thus (2.9) would be a new lifting to E_2 with $f_2^*(k^{2'}_{16n+8})$ changed. A similar computation is required to determine whether (2.9) changes each of the other k-invariants, and the result is as claimed in the previous paragraph.

Similarly, varying through K_{16n+3} changes $f_2^*(k^2_{16n+4})$, $f_2^*(k^2_{16n+8})$, $f_2^*(k^2_{16n+9})$, and $f_2^*(k^{2'}_{16n+10})$. These occur because the following terms, respectively, are nonzero:

$\mathrm{Sq}^1 x^{16n+3}$, $\mathrm{Sq}^2 \mathrm{Sq}^3(x^{16n+3})$, $w_4 \mathrm{Sq}^2(x^{16n+3})$, and $w_4 \mathrm{Sq}^3(x^{16n+3})$. The only other way in which $f_2^*(k_{16n+\delta}^2)$ can be changed is by varying through K_{16n+7}, which changes $f_2^*(k_{16n+8}^2)$ and $f_2^*(k_{16n+10}^2)$.

Thus varying the lifting through F_1 in such a way that both $f_2^*(k_{16n+4}^2)$ and $f_2^*(k_{16n+8}^2)$ are changed to become 0 will cause either $f_2^*(k_{16n+10}^2)$ or $f_2^*(k_{16n+10}^{2'})$ to become nonzero. Thus if $f_1 : P^{16n+10} \to E_1$ represents a lifting obtained by factoring through HP^{4n+2}, then every lifting of f_1 to E_2 sends some k-invariants nontrivially, and hence f_1 does not lift to $\widetilde{BSp}(16n + 1)$.

We must also consider the possibility that f_1 could be varied through the fiber, F_0, of the map $E_1 \to BSp$ in such a way that the new map f_1' lifted to f_2' : $P^{16n+10} \to E_2$ and sent all k^2-invariants to 0. This is the secondary indeterminacy consideration that had led the first author to not try to prove nonimmersions by ordinary MPTs in his work during the 1970s and 1980s. However, in the case at hand, secondary indeterminacy is not a problem because any nontrivial map $P^{16n+10} \to F_0$ will change the images of some of the k^1-invariants. Since f_1 was a map which lifted to E_2 and therefore sent all the k^1-invariants to 0, any variation of f_1 through F_0 will not lift to E_2.

This completes the proof that $(32n + 12)\xi_{16n+10}$ does not lift to $\widetilde{BSp}(16n + 1)$ when $\alpha(n) = 2$, once we verify the statement made in the preceding paragraph about varying maps through F_0. Varying through K_{16n+1} changes $f_1^*(k_{16n+4}^1)$, $f_1^*(k_{16n+5}^1)$, and $f_1^*(k_{16n+9}^{1'})$. Varying through K_{16n+3} changes $f_1^*(k_{16n+4}^1)$, $f_1^*(k_{16n+7}^1)$, $f_1^*(k_{16n+8}^1)$, and $f_1^*(k_{16n+9}^{1'})$. Varying through K_{16n+7} changes $f_1^*(k_{16n+8}^1)$ and $f_1^*(k_{16n+9}^1)$. Any nontrivial combination of these changes some $f_1^*(k_{16n+\delta}^1)$, as claimed.

3. Proof of embeddings

In this section, we prove Theorem 1.2. We use the following result of Mahowald, which deals with topological embeddings.

Theorem 3.1 ([8]). *Assume that P^q embeds in $\mathbf{R}^p$ with normal bundle ν.*

- *If $\nu \otimes \xi_q$ has n linearly independent sections and P^{n-1} embeds in S^{m-1}, then P^{n+q} embeds in $\mathbf{R}^{p+m}$.*
- $\nu \otimes \xi_q \oplus (q + 1)\epsilon \approx (p + 1)\xi_q.$

We apply this result to the embedding of P^{8n+2} in $\mathbf{R}^{16n-1}$ when $\alpha(n) > 2$, as proved in [11]. Using also the embedding of P^1 in S^1, we obtain Theorem 1.2 once we prove the following result. Here $\theta = \nu \otimes \xi_{8n+2}$.

Theorem 3.2. *If θ is an $(8n - 3)$-plane bundle over P^{8n+2} which is stably equivalent to $16n\xi_{8n+2}$, and $\alpha(n) > 2$, then θ has at least 2 linearly independent sections.*

Theorem 3.1 will then imply that there is a topological embedding of P^{8n+4} in $\mathbf{R}^{16n+1}$. Such an embedding can be approximated by a differentiable embedding by a result of Haefliger ([7]), since $2(16n + 1) \geq 3(8n + 4)$.

The first step toward proving Theorem 3.2 is to prove the following lemma.

Lemma 3.3. *If $\alpha(n) > 2$, the map $P^{8n+2} \to BSp$ which classifies $16n\xi_{8n+2}$ lifts to $\widetilde{BSp}(8n - 5)$.*

Proof. Similar to the work of the previous section, we consider the following diagram.

$$\begin{array}{ccc}
P_{8n-5} & \xrightarrow{\ i\ } & P_{8n-5} \wedge bo \\
\downarrow & & \downarrow \\
\widetilde{BSp}(8n-5) & \xrightarrow{\ j\ } & B^o(8n-5) \\
\downarrow & & \downarrow \\
P^{8n+2} \to HP^{2n} \xrightarrow{4nH} \quad BSp & \xrightarrow{\ =\ } & BSp
\end{array}$$

The morphism $\pi_*(P_{8n-5}) \to ko_*(P_{8n-5})$ of homotopy groups of fibers is depicted in the following ASS charts. Again, the first is from [9] while the second is well known, e.g. [4]. The morphism is easily seen to be surjective.

$$\pi_*(P_{8n-5}) \qquad\qquad\qquad ko_*(P_{8n-5})$$

$$* = \quad 8n-5 \qquad 8n-1 \qquad\qquad 8n-5 \qquad 8n-1$$
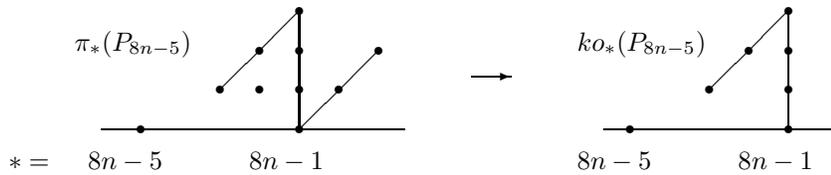
From the chart, we see that $\nu(|ko_{8n-5}(P_{8n-5})|) = 1$, and $\nu(|ko_{8n-1}(P_{8n-5})|) = 4$. One easily calculates $\nu(\binom{4n}{2n-1}) > 2$ and $\nu(\binom{4n}{2n}) = \alpha(n)$. If $\alpha(n) > 3$, then, by Theorem 2.6, $4nH_{2n}$ lifts to a map $\ell : HP^{2n} \to B^o(8n-5)$. Let $F = \text{fiber}(j) = \text{fiber}(i)$. Since $\pi_*(F) = 0$ when $* \equiv 3 \bmod 4$ and $* \leq 8n+2$, ℓ pulls back to the desired map $HP^{2n} \to \widetilde{BSp}(8n-5)$.

When $\alpha(n) = 3$, we can show, similar to the proof of Theorem 2.7, that $4nH_{2n}$ lifts to $E_3^o(8n-5)$, the third stage of the MPT for $B^o(8n-5)$. This uses Theorem 2.6 to see that $4nH_{2n}$ lifts to $B^o(8n-3)$ and $4nH_{2n-1}$ lifts to $B^o(8n-5)$, which imply that the obstructions to lifting to $E_3^o(8n-5)$ map trivially. The map $HP^{2n} \to E_3^o(8n-5)$ pulls back to E_3, the third stage in the MPT for $\widetilde{BSp}(8n-5)$, since $H^*(HP^{2n}; \pi_{*-1}(\text{fiber})) = 0$.

The fiber of $E_3 \to E_2$ contains a K_{8n-1}-factor, corresponding to the third dot up in the ASS chart for $\pi_{8n-1}(P_{8n-5})$. The map $f_3 : RP^{8n+2} \to HP^{2n} \to E_3$ can be varied through this factor to change $f_3^*(k_{8n}^3)$. This is implied by a $\text{Sq}^1\, k_{8n}^2$ which appears in the relation for k_{8n}^3 in the appropriate MPT. Since k_{8n}^3 is the only level-3 k-invariant, we deduce that there is a choice of the lifting f_3 which lifts to $E_4 = \widetilde{BSp}(8n-5)$. $\qquad\square$

The bundle θ with which we ultimately must deal might not be a symplectic bundle; however, it certainly is a Spin bundle. We reinterpret Lemma 3.3 to say that the map $16n\xi : P^{8n+2} \to \text{BSpin}$ lifts to $\text{BSpin}(8n-5)$. Next we consider the following diagram.

$$(3.4) \qquad \begin{array}{ccc}
V_{8n-3,2} & \to & P_{8n-5} \\
\downarrow & & \downarrow \\
\text{BSpin}(8n-5) & \xrightarrow{\ =\ } & \text{BSpin}(8n-5) \\
\downarrow & & \downarrow \\
P^{8n+2} \xrightarrow{\ \theta\ } \text{BSpin}(8n-3) & \to & \text{BSpin}
\end{array}$$

The first fiber, $V_{8n-3,2}$, is a Stiefel manifold, and in the stable range is homotopy equivalent to the stunted real projective space P_{8n-5}^{8n-4}. The induced morphism of homotopy groups of fibers can be easily determined using the tables of [9] to be as below, where the big dots map across. It is useful to note here that these two charts fit into an exact sequence in which the third chart is $\pi_*(P_{8n-3})$.
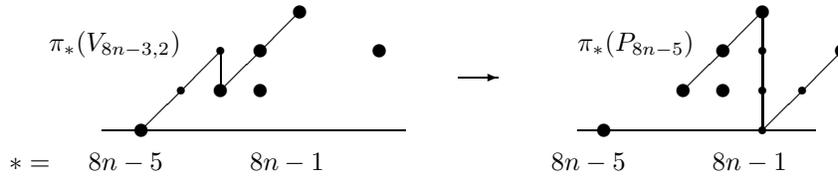

FIGURE 3.5.

The MPT for the fibration $\mathrm{BSpin}(8n-5) \to \mathrm{BSpin}(8n-3)$ corresponds to the above chart of $\pi_*(V_{8n-3,2})$. We denote its spaces by A_i.

$$\mathrm{BSpin}(8n-5)$$

$$K_{8n-3} \times K_{8n-2} \times K_{8n+1} \searrow$$
$$A_3 \to K_{8n}$$

$$K_{8n-4} \times K_{8n-3} \times K_{8n-2} \searrow$$
$$A_2 \to K_{8n-2} \times K_{8n-1} \times K_{8n+2}$$

$$K_{8n-5} \searrow$$
$$A_1 \to K_{8n-3} \times K_{8n-2} \times K_{8n-1}$$

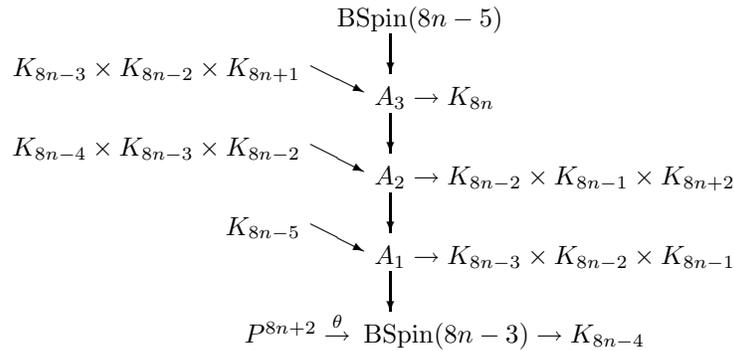$$P^{8n+2} \xrightarrow{\theta} \mathrm{BSpin}(8n-3) \to K_{8n-4}$$

FIGURE 3.6.

We need to know the relations that give rise to these k-invariants. We list them without listing the w_4 and w_8 which appear in some of these relations, because $w_4 = 0$ and $w_8 = 0$ for the bundle being considered here.

w_{8n-4}
$k_{8n-3}^1 : \mathrm{Sq}^2 w_{8n-4}$
$k_{8n-2}^1 : \mathrm{Sq}^2 \mathrm{Sq}^1 w_{8n-4}$
$k_{8n-1}^1 : \mathrm{Sq}^4 w_{8n-4}$
$k_{8n-2}^2 : \mathrm{Sq}^1 k_{8n-2}^1 + \mathrm{Sq}^2 k_{8n-3}^1$
$k_{8n-1}^2 : \mathrm{Sq}^2 k_{8n-2}^1$
$k_{8n+2}^2 : (\mathrm{Sq}^4 + \mathrm{Sq}^3 \mathrm{Sq}^1)k_{8n-1}^1 + \mathrm{Sq}^6 k_{8n-3}^1$
$k_{8n}^3 : \mathrm{Sq}^3 k_{8n-2}^2 + \mathrm{Sq}^2 k_{8n-1}^2$

Primary indeterminacy allows the following classes to be changed (if liftings to the indicated level exist):

- $f_1^*(k_{8n-3}^1)$ (through K_{8n-5});
- $f_2^*(k_{8n-2}^2)$ (through K_{8n-3});
- $f_2^*(k_{8n+2}^2)$ (through K_{8n+2});
- $f_3^*(k_{8n}^3)$ (through K_{8n}).

We can also use relations in the MPT to deduce that (again, if the liftings to the indicated level exist) $f_1^*(k_{8n-2}^1) = 0$ and $f_2^*(k_{8n-1}^2) = 0$. To see the first of these, we use the relation that gives rise to k_{8n-1}^2. This says that $\mathrm{Sq}^2 k_{8n-2}^1 = 0 \in H^*A_1$. Thus $\mathrm{Sq}^2(f_1^*(k_{8n-2}^1)) = 0 \in H^*(P^{8n+2})$, and so we cannot have $f_1^*(k_{8n-2}^1) = x^{8n-2}$. The equation $f_2^*(k_{8n-1}^2) = 0 \in H^*A_2$ follows similarly from the relation for k_{8n}^3.

Because $w_{8n-4}(\theta) = w_{8n-4}(16n\xi) = \binom{16n}{8n-4}x^{8n-4} = 0 \in H^*(P^{8n+2})$, the map θ lifts to $f_1 : P^{8n+2} \to A_1$. Once we show that $f_1^*(k_{8n-1}^1) = 0$, we will be able to deduce our desired lifting to $\mathrm{BSpin}(8n-5)$, for all other k-invariants are either in the indeterminacy or map to 0 by relations, as described in the preceding paragraph.

The map of fibrations in (3.4) induces a map of MPTs. The element in filtration 1 in π_{8n-2}(fibers) maps across, as indicated in Figure 3.5. This implies that the corresponding level-1 k-invariant maps across; i.e., if $g_1 : A_1 \to E_1$ is the induced map of spaces in the MPTs, then $g_1^*(\tilde{k}_{8n-1}^1) = k_{8n-1}^1$, where we use the tilde to indicate the k-invariants in the MPT for $\mathrm{BSpin}(8n-5) \to \mathrm{BSpin}$. In the paragraph after the proof of Lemma 3.3, we have observed that the map θ into BSpin lifts to $\mathrm{BSpin}(8n-5)$. This implies that the composite $P^{8n-2} \xrightarrow{f_1} A_1 \xrightarrow{g_1} E_1$ differs from a map h_1 that lifts to $\mathrm{BSpin}(8n-5)$ by a map δ_1 which factors through the fiber $K_{8n-5} \times K_{8n-1}$ of $E_1 \to \mathrm{BSpin}$. The relation for $\tilde{k}_{8n-1}^1$ is $(\mathrm{Sq}^4 + w_4)w_{8n-4}$. Since $\mathrm{Sq}^4(x^{8n-5}) = 0$ and $w_4(\theta) = 0$, any such map δ_1 must send $\tilde{k}_{8n-1}^1$ to 0. Since h_1 lifts, it sends all k-invariants to 0. Thus

$$0 = h_1^*(\tilde{k}_{8n-1}^1) + \delta_1^*(\tilde{k}_{8n-1}^1) = f_1^* g_1^*(\tilde{k}_{8n-1}^1) = f_1^*(k_{8n-1}^1),$$

as desired.

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