ON THE DENSITY OF THE SET OF GENERATORS
OF A POLYNOMIAL ALGEBRA

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Abstract. Let \( K[X] = K[x_1,\ldots,x_n], \quad n \geq 2, \) be the polynomial algebra over a field \( K \) of characteristic 0. We call a polynomial \( p \in K[X] \) coordinate (or a generator) if \( K[X] = K[p,p_2,\ldots,p_n] \) for some polynomials \( p_2,\ldots,p_n \). In this note, we give a simple proof of the following interesting fact: for any polynomial \( h \) of the form \( (x_i + q) \), where \( q \) is a polynomial without constant and linear terms, and for any integer \( m \geq 2 \), there is a coordinate polynomial \( p \) such that the polynomial \( (p - h) \) has no monomials of degree \( \leq m \). A similar result is valid for coordinate \( k \)-tuples of polynomials, for any \( k < n \). This contrasts sharply with the situation in other algebraic systems.

On the other hand, we establish (in the two-variable case) a result related to a different kind of density. Namely, we show that given a non-coordinate two-variable polynomial, any sufficiently small perturbation of its non-zero coefficients gives another non-coordinate polynomial.

1. Introduction

Let \( K[X] = K[x_1,\ldots,x_n], \quad n \geq 2, \) be the polynomial algebra over a field \( K \) of characteristic 0. We denote by \( \text{mindeg}(p) \) the minimal degree of non-zero monomials of \( p \in K[X] \).

We call automorphic images of \( x_1 \) coordinate polynomials to simplify the language. Similarly, a \( k \)-tuple of polynomials \( (p_1,\ldots,p_k), \quad p_i \in K[X], \quad k \leq n, \) is coordinate if there exists an automorphism of \( K[X] \) which sends \( x_1,\ldots,x_k \) respectively to \( p_1,\ldots,p_k \). Equivalently, a \( k \)-tuple \( (p_1,\ldots,p_k) \) is coordinate if there are polynomials \( p_{k+1},\ldots,p_n \in K[X] \) such that \( K[p_1,\ldots,p_k,p_{k+1},\ldots,p_n] = K[X] \).

In this note, we give a simple proof of the following interesting fact: the set of coordinate polynomials is dense (in the formal power series topology) in the set of polynomials of the form \( (x_i + q), \) mindeg\((q) \geq 2 \). That is, any polynomial of this form can be completed to a coordinate polynomial by monomials of arbitrarily high degree. Actually, our proof yields a somewhat stronger result:

Theorem 1.1. For any \( (n - 1) \)-tuple of polynomials \( (h_1,\ldots,h_{n-1}) \) of the form \( h_i = x_i + r_i, \) mindeg\((r_i) \geq 2, \) \( i = 1,\ldots,n - 1, \) and any integer \( m \geq 2, \) there
exists a coordinate \((n - 1)\)-tuple \((p_1, \ldots, p_{n-1})\) such that \(\text{mindeg}(p_i - h_i) > m, i = 1, \ldots, n - 1\).

There is a non-commutative version of Theorem 1.1 (see [4]) which involves machinery from representation theory of the general linear group \(GL_n(K)\). Our proof here is based on simpler ideas and is a consequence of a result of Anick [1].

Theorem 1.1 contrasts sharply with the situation in other, non-commutative, algebras. For example, the set of primitive elements (that is what generators are usually called in a non-commutative setting) of a free Lie algebra of rank 2 is not dense because by a theorem of Cohn [2], all automorphisms of this algebra are linear. Moreover, although the automorphism groups of \(K[x_1,x_2]\) and \(K\langle x_1, x_2 \rangle\) (the free associative algebra of rank 2) are isomorphic (see e.g. [3]), we have:

**Proposition 1.2.** The element \(u = x_1 + x_1x_2\) cannot be completed to a primitive element of \(K\langle x_1, x_2 \rangle\) by monomials of degree higher than 2.

The proof of Proposition 1.2 is based on a characterization of generators of \(K\langle x_1, x_2 \rangle\) as it appears in [6].

Finally, we establish (in the two-variable case) a result related to a different kind of density:

**Theorem 1.3.** Let \(p(x,y) = \sum_{i,j=1}^{m} c_{ij} \cdot x^i y^j, c_{ij} \in K,\) be a non-coordinate polynomial from \(K[x,y]\). Let \(K = \mathbb{R}\) or \(\mathbb{C}\). Then there is an \(\varepsilon > 0\) such that every polynomial \(q(x,y) = \sum_{i,j=1}^{m} c'_{ij} \cdot x^i y^j\) with \(|c_{ij} - c'_{ij}| < \varepsilon\) if \(c_{ij} \neq 0\) and \(c'_{ij} = 0\) if \(c_{ij} = 0\), is non-coordinate as well.

2. Preliminaries

For background on polynomial automorphisms we refer to the book [3]. Anick [1] proved that, with respect to the formal power series topology, the set \(J\) of endomorphisms of \(K[X]\) with an invertible Jacobian matrix is a closed set, and the group of tame automorphisms of \(K[X]\) is dense in \(J\). This means that for any polynomial mapping \(F = (f_1, \ldots, f_n)\) with invertible Jacobian matrix \(J(F)\) (i.e. \(0 \neq J(F) \in K\)) and for any positive integer \(m\), there is a tame automorphism \(G = (g_1, \ldots, g_n)\) such that the polynomials \(f_i - g_i\) contain no monomials of degree less than \(m\). An interpretation of the result of Anick in the language we need is given in [5] Theorem 4.2.7. We recall some details here briefly.

Let \(P_k\) be the \(K\)-vector space of all homogeneous polynomials of degree \(k \geq 0\). Let \(I_k, k \geq 2,\) be the semigroup of all polynomial endomorphisms \(F = (f_1, \ldots, f_n)\) such that \(x_i\) is the only monomial of \(f_i\) of degree less than \(k;\) \(i = 1, \ldots, n\). We write

\[ f_i = x_i + g_i + h_i, \]

where \(g_i \in P_k\) is the homogeneous component of \(f_i\) of degree \(k,\) and \(\text{mindeg}(h_i) > k.\)

It turns out that there is a homomorphism \(\phi\) of \(I_k\) onto the direct sum of additive groups \(P_k^{\oplus n} \cong P_k \oplus \ldots \oplus P_k\) such that \(\phi(F) = (g_1, \ldots, g_n).\)

Let \(T\) be the group of tame automorphisms of the algebra \(K[X],\) and let \(S_k\) be the set of all polynomial mappings \(S = (s_1, \ldots, s_n) \in I_k\) such that \(s_i = x_i + g_i + h_i,\) where \(g_i \in P_k\) and \(h_i \in \sum_{l>k} P_l,\) with the property

\[ \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} = 0.\]
The main step of Anick’s proof (see [5], Step 2 of the proof of Theorem 4.2.7) is to show that

$$\phi(T \cap I_k) = \phi(S_k).$$

This yields

**Lemma 2.1.** For any $S \in S_k \subseteq I_k$, there is a tame automorphism $G_k$ of $K[X]$ such that $S \circ G_k^{-1}$ is in $I_{k+1}$.

3. **Proofs**

**Proof of Theorem 1.1.** Let $u_1, \ldots, u_{n-1}$ be $n - 1$ polynomials without constant and linear terms, and let $u_{ij}$ be the homogeneous component of degree $j$ of the polynomial $u_i$; $j = 2, \ldots, m$. Let $k$ be the smallest integer such that $u_{ik} \neq 0$ for some $i$. Let, for example, $i = 1$.

Write the partial derivative of $u_{1k}$ with respect to $x_1$ in the form

$$\frac{\partial u_{1k}}{\partial x_1} = \sum_{j=0}^{k-1} a_j x_n^j,$$

where the polynomials $a_j$ do not depend on $x_n$. There is a homogeneous polynomial $w_{1k} \in P_k$ such that

$$\frac{\partial w_{1k}}{\partial x_2} = -\sum_{j=0}^{k-1} a_j x_n^j.$$

Consider an endomorphism $F_{1k}$ of the algebra $K[X]$ defined by

$$F_{1k} = (x_1 + u_{1k}, x_2, \ldots, x_{n-1}, x_n + w_{1k}).$$

Clearly, $F_{1k} \in I_k$ and, because of the choice of $w_{1k}$, also $F_{1k} \in S_k$. Similarly, we construct endomorphisms

$$F_{ik} = (x_1, \ldots, x_{i-1}, x_i + u_{ik}, x_{i+1}, \ldots, x_{n-1}, x_n + w_{ik})$$

for some $w_{ik} \in P_k$, $i = 2, \ldots, n - 1$, such that $F_{ik} \in S_k$. Hence the composition $F_k = F_{1,k} \circ \cdots \circ F_{n-1,k}$ also belongs to $S_k$ and by Lemma 2.1 there exists a tame automorphism $G_k = (g_{1,k}, \ldots, g_{n,k}) \in I_k$ such that $F_{k+1} = F_k \circ G_k^{-1} \in I_{k+1}$. Therefore, $g_{i,k} = x_i + u_{i,k} + v_{i,k+1} + v_i$, $i = 1, \ldots, n - 1$, where $v_{i,k+1}$ is homogeneous of degree $k + 1$, and $v_i \in \sum_{l \geq k+2} P_l$.

Continuing this way, we obtain a tame automorphism $G_{k+1} \in I_{k+1}$ such that $g_{i,k+1} = x_i + (u_{i,k+1} - v_{i,k+1}) + w_i$, $i = 1, \ldots, n - 1$, where $w_i \in \sum_{l \geq k+2} P_l$.

If we act by the automorphism $G_{k+1}$ on the polynomial $u_{ik}$, we get a polynomial of the form $u_{ik} + s_i$, where $s_i$ has no homogeneous components of degree less than $(k - 1) + (k + 1) = 2k > k + 1$. Therefore, the automorphism $G_{k+1} \circ G_k$ takes $x_i$ to $x_i + u_{ik} + u_{i,k+1}$ (terms of higher degree), $i = 1, \ldots, n - 1$.

In a finite number of steps, we obtain a tame automorphism $G = (g_1, \ldots, g_n)$ such that

$$g_i = x_i + u_i + \text{(terms of higher degree)}, \quad i = 1, \ldots, n - 1.$$

This completes the proof of Theorem 1.1. 

\[\square\]
Proof of Proposition 1.2. By way of contradiction, suppose there is an element \( w \) without monomials of degree lower than 3, such that \( u = x_1 + x_1x_2 + w \) is a primitive element of \( K\langle x_1, x_2 \rangle \).

By Corollary 1.5 of [3], every primitive element of \( K\langle x_1, x_2 \rangle \) is palindromic, i.e., is invariant under the operator \( \pi \) that re-writes every monomial backwards. For example, \( (x_1x_2)^\pi = x_2x_1; (x_1x_2x_1x_2^2)^\pi = x_2^3x_1x_2x_1, \) etc.

It is clear that if an element of \( K\langle x_1, x_2 \rangle \) is palindromic, then its every homogeneous component is palindromic, too. Since the homogeneous component of degree 2 of our element \( u \) is not palindromic, this yields a contradiction.

Remark. Clearly, the statement of Proposition 1.2 holds for any element \( u = x_1 + a \cdot x_1x_2 + b \cdot x_2x_1 \in K\langle x_1, x_2 \rangle \), where \( a, b \in K \) and \( a \neq b \).

A combination of Theorem 1.1 and Proposition 1.2 calls for an example of a coordinate polynomial \( p \in K[x_1, x_2] \) of the form \( x_1 + x_1x_2 + \) (terms of higher degree); an example like that is given below.

Example. The polynomial

\[
p = x_1 + x_1x_2 + \frac{1}{4}(x_1x_2^2 - x_1^2x_2 - x_1^3 + x_2^3) - \frac{1}{16}(x_1 + x_2)^4
\]

is coordinate since it is the image of \( x_1 \) under the automorphism \( \alpha \beta \alpha^{-2} \beta \alpha \), where \( \alpha \) takes \( x_1 \) to \( (x_1 + x_2) \) and fixes \( x_2 \), and \( \beta \) fixes \( x_1 \) and takes \( x_2 \) to \( (x_2 - \frac{x_1^2}{4}) \).

Proof of Theorem 1.3. Here we use a characterization of two-variable polynomial automorphisms given in [3, Theorem 6.8.5], which implies, in particular, that if \( p(x, y) \in K[x, y] \) is a (non-linear) coordinate polynomial, then there is an elementary automorphism of the form \( \{ x \to x + \lambda \cdot y^m; y \to y \} \) or \( \{ x \to x; y \to y + \lambda \cdot x^m \} \), \( \lambda \in K^* \), that decreases the degree of \( p(x, y) \).

Let \( p(x, y) = \sum c_{ij} \cdot x^iy^j \), \( c_{ij} \in K^* \), be a non-linear polynomial. Then the condition in the previous paragraph translates into a system of homogeneous polynomial equations, where \( c_{ij} \) are considered indeterminates, and (polynomial) functions of \( \lambda \) are considered coefficients (every equation in this system expresses the condition on the coefficient at a particular monomial to be equal to zero). Thus, the set of solutions of this system is a subset of \( K^* \), where \( s \) is the number of non-zero coefficients of our polynomial \( p(x, y) \).

If \( p(x, y) \) is a coordinate polynomial, then this system has non-zero solutions. If it is not, then the system might or might not have non-zero solutions. If it does not, then, since the set of non-solutions of a polynomial system is an open set (because it is the complement of a closed set), the result follows. Note that the presence of "unfit" solutions (where some of \( c_{ij} \) are equal to zero) does not change the openness of the set of non-solutions since it is equivalent to adding to this set several sets of a smaller dimension, and those are always closed sets.

If our system has non-zero solutions, then we apply an elementary automorphism to the polynomial \( p(x, y) \) and reduce its degree. The new polynomial is still non-coordinate, and its coefficients are polynomial functions of \( c_{ij} \) and \( \lambda \). Now applying the same argument to this new polynomial yields the result because continuing the reduction of the degree, we eventually obtain a system without non-zero solutions.

\[ \square \]
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