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$\mathcal{R}$-COMPACT SPACES WITH weight $X < \mathrm{Exp}_{\mathcal{R}} X$

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Abstract. Answering a question of Arhangel'skii, we show – under GCH – that for most cardinals $\mathfrak{m}$ there exists an $\mathcal{R}$-compact space X such that weight $X = \mathfrak{m}$ but X does not embed in a closed fashion into the product of $\mathfrak{m}$ copies of $\mathcal{R}$.

1. Introduction

All spaces are assumed to be Hausdorff completely regular. For spaces X and Y we write $X \underset{\text{top}}{\subset} Y$ ($X \underset{\text{cl}}{\subset} Y$) provided that X is homeomorphic to a subspace (a closed subspace) of Y. $\mathcal{N}$ and $\mathcal{R}$ stand for the space of the positive integers and the reals, respectively.

Following [M1], we let, for an E-completely regular (E-compact) space X, $\exp_E X$ ($\mathrm{Exp}_E X$) to be the smallest infinite cardinal $\mathfrak{m}$ so that $X \underset{\text{top}}{\subset} E^{\mathfrak{m}}$ ($X \underset{\text{cl}}{\subset} E^{\mathfrak{m}}$). $\exp_E X$ is usually very simple to determine; for small E, in particular, for $E = \mathcal{N}$ and $E = R$, we have $\exp_E X =$ weight X (of course, if E is large, then $\exp_E X$ can be much smaller than weight X).

Determination of $\mathrm{Exp}_E X$ is more intricate. Obviously, we have $\exp_E X \leq \mathrm{Exp}_E X$. Equality need not hold: A trivial example, mentioned in [M2], is the space $\mathcal{Q}$ of the rationals: we have $\exp_{\mathcal{N}} \mathcal{Q} = \exp_{\mathcal{R}} \mathcal{Q} = \aleph_0 < \mathrm{Exp}_{\mathcal{N}} \mathcal{Q} = \mathrm{Exp}_{\mathcal{R}} \mathcal{Q}$. Other examples have been found (see section 4 of this paper), but only among 2nd countable spaces. But this is not enough to justify the notations $\mathrm{Exp}_E X$ versus $\exp_E X$. Such justification calls for the evidence that the failure of the equality $\exp_E X = \mathrm{Exp}_E X$ is a quite common occurrence or that, at least, it happens for spaces with higher $\exp_E X$. Apparently, I have considered this to be obvious (more precisely, I did not realize that this might present any difficulty) and I did not make any efforts in this direction.

Well, this is not exactly obvious. In early December 1996, I received an e-mail message from my dear and long-unseen friend Arhangel'skii asking (rough quote): "is it true that for $\mathcal{R}$-compact space X with weight $X = 2^{\aleph_0}$ we have $\mathrm{Exp}_{\mathcal{R}} X = 2^{\aleph_0}$?" and "what about the higher cardinals?" To my embarassment, the best I could do at that time was to give an evasive answer: "I do not think so, but I will have to refresh my memory." This indeed is true but it took a bit more than just a

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memory refresh: using Hausdorff η_α-sets,[1] the situation concerning $\operatorname{Exp}_{\mathcal{R}}\mathcal{Q}$ can be reproduced in higher cardinalities. But this requires the Generalized Continuum Hypothesis and works only for regular Ulam non-measurable cardinals. Thus the matters are just opened rather than settled. In fact, one of the merits of this paper is that it brings a long-belated recognition that the study of the large exponent is a large and practically unexplored area of investigation. More concrete discussion given in section 4.D.

Concluding this discussion, I wish to express my sincere gratitude to the referee and to Professor A. Dow for their thoughtful suggestions.

Now, the result. Let $\mathfrak{m}_U$ be the first (uncountable) Ulam measurable cardinal.

1.1. Theorem. *If $\mathfrak{m} = \mathfrak{n}^+ = 2^{\mathfrak{n}} < \mathfrak{m}_U$, then there exists a hereditarily $\mathcal{N}$-compact space X with* $\operatorname{card} X = \operatorname{weight} X = \exp_{\mathcal{R}} X = \mathfrak{m} < \operatorname{Exp}_{\mathcal{R}} X$.

More precisely, this X fulfills $\operatorname{Exp}_R X > \mathfrak{m}$ in a strong way: X does not admit a continuous one-to-one mapping onto a closed subspace of a product of $\mathfrak{m}$ spaces whose pseudocharacters are bounded below $\mathfrak{m}$ (i.e., there is an $\mathfrak{m}_1 < \mathfrak{m}$ such that the pseudocharacter of each of these spaces is $\leq \mathfrak{m}_1$).

If we remove "hereditarily", then the first part of the above can be stated entirely in terms of βX (without any reference to closed embedding):

1.1a. *If $\mathfrak{m} = \mathfrak{n}^+ = 2^{\mathfrak{n}} < \mathfrak{m}_U$, then there exists a space X with* $\operatorname{card} X = \operatorname{weight} X = \mathfrak{m}$ *and such that $\beta X \setminus X$ can be written as the union of CO_δ-subsets of βX, but any such union must contain $> \mathfrak{m}$ terms, even if these terms are required to be only functionally closed in βX.*

For the proof of the equivalence see the last sentence in section 4.A. (A CO_δ-set is a set which is the intersection of countably many clopen sets.)

2. Definition and the basic property of $\mathcal{Q}(\mathfrak{m})$

We will be using spaces – studied first by Sikorski [Sik] – in which closure is additive in higher cardinalities. Thus, if $\mathfrak{m}$ is a cardinal, then an $\mathfrak{m}$-space (or a space with $\mathfrak{m}$-topology) is a space in which the equality $\overline{\bigcup \mathfrak{R}} = \bigcup \{\overline{A} : A \in \mathfrak{R}\}$ holds for every class $\mathfrak{R}$ of subsets with $\operatorname{card} \mathfrak{R} < \mathfrak{m}$. The smallest $\mathfrak{m}$-topology containing a given topology τ will be denoted as $\tau * \mathfrak{m}$. If $\mathfrak{B}$ is a base for τ, then the class of all intersections of fewer than $\mathfrak{m}$ members of $\mathfrak{B}$ is a base for $\tau * \mathfrak{m}$.

Elements of a product of $\mathfrak{m}$ spaces will be treated as functions on the ordinals $< \omega_\alpha$, where $\overline{\omega}_\alpha = \mathfrak{m}$; thus, if x is an element of such product and ξ, ξ' are ordinals $< \omega_\alpha$, then $x(\xi)$ is the ξ-th coordinate of x (on the other hand, x_ξ and $x_{\xi'}$ denote two points of the product rather than two coordinates of the same point). $x|\xi$ is the restriction of x to the ordinals $< \xi$.

Now, let $\widehat{\mathcal{D}^{\mathfrak{m}}}$ be the product of $\mathfrak{m}$ copies of the two-point discrete space $\mathcal{D} = \{0, 1\}$ with the topology $\tau_\Pi * \mathfrak{m}$, where τ_Π is the usual product topology in $\mathcal{D}^{\mathfrak{m}}$. Let $\mathcal{Q}(\mathfrak{m})$ be the subspace of $\widehat{\mathcal{D}^{\mathfrak{m}}}$ consisting of all $x \in \mathcal{D}^{\mathfrak{m}}$ such that $\operatorname{card}\{\xi : x(\xi) \neq 0\} < \mathfrak{m}$. (Note that under the lexicographic order $\mathcal{Q}(\mathfrak{m})$, with $\mathfrak{m} = \aleph_\alpha$, is an η_α-set and the topology is the order topology.)

[1]The familiarity with η_α-sets is needed only in section 4.B. In section 2 we give an explicit definition through a product construction, where, to emphasize the analogy with the space of rationals, we use the symbol $\mathcal{Q}(\mathfrak{m})$.

Hereditary $\mathcal{N}$-compactness of $\mathcal{Q}(\mathfrak{m})$ follows from general theorems on stability of E-compactness and will be given in the next section. At this time we will prove that $\mathcal{Q}(\mathfrak{m})$ satisfies the second part of Theorem 1.1. The argument is an adaptation of the Cantor proof of uncountability of the continuum.

Proof. Let Z be the product mentioned in the second part of Theorem 1.1 and let φ be a continuous one-to-one map of $X = \mathcal{Q}(\mathfrak{m})$ onto a closed subspace Y of Z. Let $\overline{\omega}_\alpha = \mathfrak{m}$ and let $y_0, y_1, \ldots, y_\xi, \ldots;\ \xi < \omega_\alpha$, be the sequence (without repetitions) of all points of Y. The continuity of φ (stated in the classical ϵ-δ form) implies that

$$(*)\qquad \begin{array}{l} \textit{for every } x \in X \textit{ and every } \epsilon < \omega_\alpha \textit{ there is a } \delta < \omega_\alpha \textit{ such that} \\ \textit{for every } x' \in X,\ x|\delta = x'|\delta \textit{ implies } \varphi(x)|\epsilon = \varphi(x')|\epsilon. \end{array}$$

We shall define a sequence x_ξ of points of X and two increasing sequences ϵ_ξ and δ_ξ of ordinals such that for every $\xi' < \xi$

$$\text{(a)}\ \ x_\xi|\delta_{\xi'} = x_{\xi'}|\delta_{\xi'}, \qquad \text{(b)}\ \ \varphi(x_\xi)|\epsilon_\xi \neq y_\xi|\epsilon_\xi,$$
$$\text{(c)}\ \ \text{for every } x' \in X, \quad x'|\delta_\xi = x_\xi|\delta_\xi \ \text{ implies } \ \varphi(x')|\epsilon_\xi = \varphi(x_\xi)|\epsilon_\xi.$$

Let x_0 be any point with $\varphi(x_0) \neq y_0$, let ϵ_0 be such that $\varphi(x_0)|\epsilon_0 \neq y_0|\epsilon_0$; δ_0 is selected according to (*). Assuming that everything is defined for all $\xi <$ a given ξ_0, we can, by (a) and the fact that X contains all sequences x with $\mathrm{card}\,\{\xi :\ x(\xi) \neq 0\} < \mathfrak{m}$, select an x_{ξ_0} so that (a) is still satisfied and $\varphi(x_{\xi_0}) \neq y_{\xi_0}$; selection of ϵ_{ξ_0} and δ_{ξ_0} is the same as in the initial step.

If $\xi' < \xi < \omega_\alpha$, then from (a) and (c) (applied to ξ') we obtain $\varphi(x_\xi)|\epsilon_{\xi'} = \varphi(x_{\xi'})|\epsilon_{\xi'}$. Consequently, there is a point $y \in Z$ such that $y|\epsilon_\xi = \varphi(x_\xi)|\epsilon_\xi$. Clearly, $y \in \overline{Y} = Y$. But, by (b), $y|\epsilon_\xi \neq y_\xi|\epsilon_\xi$. In other words, y is different from all the points of Y – a contradiction.

3. Hereditary $\mathcal{N}$-compactness of $\mathcal{Q}(\mathfrak{m})$

We have the following result on stability of E-compactness.

3.1. Theorem. (i) *If X is E-completely regular, Y is E-compact and $f : X \to Y$ is a continuous function such that* (a) *$f^{-1}(y)$ is compact for every $y \in Y$ and* (b) *$f[F]$ is E-compact for every closed $F \subset X$, then X is E-compact.*

(ii) *If $f : X \underset{\text{onto}}{\longrightarrow} Y$ is continuous, X is E-compact, Y_0 is an E-compact subspace of Y, then $f^{-1}[Y_0]$ is E-compact.*

(iii) *If $\mathcal{T}$ is a collection of E-compact topologies for a set X, then $\tau_0 = \sup \mathcal{T}$ is also E-compact.*

(iv) *The intersection of any collection of E-compact subspaces is E-compact.*

Part (i) is of some non-triviality; it is given in [M5].[2]

[2]Note that (i) is *not* a pure theorem on closed embedding; it requires some mild homogeneity conditions on E. These conditions are obviously satisfied for $E = \mathcal{R}$ and $E = \mathcal{N}$ (for details see [M5]). Since in this paper we apply (i) only for $E = \mathcal{N}$ and the proof for any 0-dimensional E is very short, it is given below.

Proof of 3.1.(i) *for* 0-*dimensional* E. Assume that X is not E-compact; let $\widehat{f} : \beta_E X \to Y$ be the continuous extension of f; let $x_0 \in \beta_E X \setminus X$, $y_0 = \widehat{f}(x_0)$, $K = f^{-1}(y_0)$. Since E, and therefore also $X \cup \{x_0\}$, is 0-dimensional and K is compact, we can find a clopen subset U of $X \cup \{x_0\}$ with $K \subset U$, $x_0 \notin U$. Since U is clopen, $\beta_E X$ is the discrete union (= topological sum) of $\beta_E U$ and $\beta_E(X \setminus U)$. Now, $x_0 \in \beta_E(X \setminus U)$, but $f[X \setminus U]$ is E-compact; therefore $f|(X \setminus U)$ extends to a continuous map of $\beta_E(X \setminus U)$ into $f[X \setminus U]$; hence $y_0 = \widehat{f}(x_0) \in f[X \setminus U]$; a contradiction.

Parts (ii), (iii) and (iv) are trivial ($f^{-1}[Y_0] \underset{\text{top}}{=} f \cap (X \times Y_0)$ and $f \cap (X \times Y_0)$ is closed in $X \times Y_0$; (X, τ_0) is homeomorphic to the diagonal of the product $\prod \{(X, \tau) : \tau \in \mathfrak{T}\}$; similar argument for (iv)) and, except for (iii), have already been stated in [M2].

From the above we obtain the following result on $\mathcal{N}$-compactness.

3.2. (i) *In an $\mathcal{N}$-compact $\mathfrak{m}$-space X, $\mathfrak{m} < \mathfrak{m}_U$, if K is the union of $\leq \mathfrak{m}$ clopen sets, then K is $\mathcal{N}$-compact.*

(ii) *If (X, τ_0) is an $\mathcal{N}$-compact $\mathfrak{m}$-space, $\mathfrak{m} < \mathfrak{m}_U$, then every topology τ with $\tau_0 \subset \tau \subset \tau_0 * \mathfrak{m}^+$ is $\mathcal{N}$-compact.*

(iii) *If (X, τ_0) is $\mathcal{N}$-compact, then for every $\mathfrak{m} \leq m_{\mathrm{U}}$, $\tau_0 * \mathfrak{m}$ is $\mathcal{N}$-compact.*

(iv) *An $\mathcal{N}$-compact $\mathfrak{m}$-space X with pseudocharacter $\psi(X) \leq \mathfrak{m}$, $\mathfrak{m} < \mathfrak{m}_U$, is hereditarily $\mathcal{N}$-compact.*

Proof. Part (i). We can assume that $K \neq X$. Since X is an $\mathfrak{m}$-space, K can be written as $\bigcup \{E_\xi : \xi \in \Xi\}$, where E_ξ are non-empty, mutually disjoint, clopen sets and $\operatorname{card} \Xi \leq \mathfrak{m}$. Letting $f(x) = \xi$ for $x \in E_\xi$ and $f(x) = \infty$ for $x \in X \setminus K$, we obtain a continuous function f from X onto the one-point compactification $\iota\Xi = \Xi \cup \{\infty\}$ of Ξ (Ξ treated as a discrete space). We have $K = f^{-1}[\Xi]$ and the conclusion follows.

Part (ii). Let f be the identity map of (X, τ) onto (X, τ_0). If F is a τ-closed subset of X, then F is the intersection of the sets of the form $X \setminus G$, where $G \in \mathbf{G}_{\mathfrak{m}}(X, \tau_0)$. In turn, each $X \setminus G$ is the union of $\leq \mathfrak{m}$ τ_0-clopen sets. By (i), $X \setminus G$ is $\mathcal{N}$-compact; thus F is $\mathcal{N}$-compact. In other words, f satisfies the condition (b) of 3.1.(i). (a) of 3.1.(i) is obviously satisfied and this ends the proof.

Part (iii). Let $\mathfrak{m}_0$ be the smallest cardinal for the above fails. By (ii), $\mathfrak{m}_0$ is a limit cardinal. Consequently, $\tau_0 * \mathfrak{m}_0 = \sup \{\tau_0 * \mathfrak{m} : \mathfrak{m} < \mathfrak{m}_0\}$. Apply 3.1.(iii).

Part (iv). The cardinality assumptions on X imply that, for every $x \in X$, $X \setminus \{x\}$ is the union of $\leq \mathfrak{m}$ clopen sets.

Note. Observe the difference between (ii) and (iii) – we do not claim that every topology between τ_0 and $\tau_0 * \mathfrak{m}_U$ is $\mathcal{N}$-compact. In fact, a topology τ with $\tau_0 \subset \tau \subset \tau_0 * \mathfrak{m}^{++}$ need not be $\mathcal{N}$-compact. For example, $X =$ the space of all ordinals $\leq \omega_1$; τ_0 is the usual order topology. This is an $\mathcal{N}$-compact $\aleph_0$-topology and $\tau_0 * \aleph_0^{++}$ is discrete. If τ is the smallest extension of τ_0 under which ω_1 is isolated, then $\tau_0 \subset \tau \subset \tau_0 * \aleph_0^{++}$ but τ is not $\mathcal{N}$-compact.

Hereditary $\mathcal{N}$-compactness of $\mathcal{Q}(\mathfrak{m})$ follows immediately from 3.2, (iii) and (iv).

4. Overview and problems

The first three subsections contain an overview of the results concerning the large exponent; problems are given in the last. Since more precise information about closed embedding are provided by the E-defect, $\operatorname{def}_E X$, rather than by $\operatorname{Exp}_E X$, the discussion will be conducted in terms of $\operatorname{def}_E X$. $\operatorname{def}_E X$ is defined (see [M1] and [M2]) to be the smallest cardinal of an E-nonextendable class for X; we have $\operatorname{Exp}_E X = \exp_E X + \operatorname{def}_E X$.

A. Background information. E-defect can be characterized in product-theoretic terms.

4.1. *For an $X \in \mathfrak{K}(E)$ the following conditions are equivalent:*

a. $\operatorname{def}_E X \leq \mathfrak{m}$;

b. $X \underset{\mathrm{cl}}{\subset} C \times E^{\mathfrak{m}}$, *where* C *is a compact space;*
c. X *admits a perfect map onto a closed subset of* $E^{\mathfrak{m}}$.

4.1 is a modification of 5.7 in [M2] and can be shown by adapting the argument from [M2]. A more comprehensive form of this statement – including more than the present 4.1 – will be given in [M5].

For a compactificaton cX of X we let $\mathfrak{d}(cX)$ ($\mathfrak{d}^*(cX)$) = the smallest cardinal $\mathfrak{d}$ so that $cX \setminus X$ is the union of $\leq \mathfrak{d}$ of CO_δ-subsets (functionally closed subsets) of cX. (CO_δ stands for the class of all countable intersections of clopen sets.)

$\mathrm{def}_{\mathcal{N}} X$ can be related to properties of compactifications of X:

4.2. $\mathrm{def}_N X$ *satisfies the following equalities:*

$$\begin{aligned}\mathrm{def}_N X =& \mathfrak{d}(\beta X) = \min\{\mathfrak{d}(cX) : cX \text{ is a compactification of } X\}\\ =& \mathfrak{d}^*(\beta_D X) = \min\{\mathfrak{d}^*(cX) : cX \text{ is a 0-dimensional compactification of } X\}.\end{aligned}$$

We have a parallel theorem for $\mathcal{R}$-defect:

4.2a. $\mathrm{def}_R X = \mathfrak{d}^*(\beta X) = \min\{\mathfrak{d}^*(cX) : cX$ *is a compactification of* $X\}$.

4.2 and 4.2a are modifications of 4.1, parts 1 and 3, from [M1]. Arguments can be adapted; for details see [M5].

The equivalence of 1.1 (with "hereditary" removed) and 1.1a follows immediately from 4.2 and 4.2a.

A concept related to E-defect was introduced in [M4]; we will describe it in a more efficient terminology. Consider the following condition on an $\mathfrak{F} \subset C(X, E)$:

($\mathrm{C_{ext}}$) : *for every superspace* X^* *of* X *with* $X \in \mathfrak{C}(E)$, *if* $\mathfrak{F} \subset C(X^*, E)|X$, *then* $C(X, E) = C(X^*, E)|X$,

where, for any class $\mathfrak{G}$ of functions on X^*, $\mathfrak{G}|X$ stands for the class of all restrictions of members $\mathfrak{G}$ to X. Define $\mathrm{ext}_E X$ to be the smallest cardinal $\mathfrak{m}$ so that there exists an $\mathfrak{F} \subset C(X, E)$ of cardinality $\leq \mathfrak{m}$ and satisfying ($\mathrm{C_{ext}}$). Plainly, $\mathrm{def}_E X \leq \mathrm{ext}_E X$; it is known (see [M3]) that the inequality can be strict even for $E = \mathcal{R}$ (and, by the same argument, also for $E = \mathcal{N}$). There are many questions concerning ext_E; in this paper, however, we shall state, at the end of subsection C, only those that are directly related to known results on $\mathrm{def}_{\mathcal{N}} X$.

B. Set-theoretic significance of Theorem 1.1. In view of the set-theoretic nature of the space $Q(\mathfrak{m})$ Theorem 1.1 can be stated in purely set-theoretic terms as a property of η_α-sets. Thus, if H_α is an η_α-set, then the set $C(H_\alpha, \mathcal{N})$ of all continuous functions from H_α to $\mathcal{N}$ is simply the set of all $\mathcal{N}$-valued functions on H_α which are constant on intervals. Let $\mathfrak{F} \subset C(H_\alpha, \mathcal{N})$. Taking into account that a subset of H_α is compact iff it is finite, it is easy to see that the parametric map $\mathbb{P}\mathfrak{F}$ generated by $\mathfrak{F}$ is a perfect map of H_α onto a closed subset of $\mathcal{N}^{\mathfrak{F}}$ iff $\mathfrak{F}$ satisfies the following condition:

(†): *for every map* $\phi : \mathfrak{F} \to \mathcal{N}$ *we have*

a) *the set* $\{x \in H_\alpha : \phi(f) = f(x)$ *for every* $f \in \mathfrak{F}\}$ *is finite;*

and

b) *if* $\mathfrak{G}$ *is a collection (possibly empty) of open intervals and for every* $x \in H_\alpha$, $f(x) = \phi(f)$ *for every* $f \in \mathfrak{F}$ *implies* $x \in \bigcup \mathfrak{G}$, *then there is a finite system* $f_1, \ldots, f_k$ *of members of* $\mathfrak{F}$ *such that* $f(x) = \phi(f_i)$ *for* $i = 1, \ldots, k$ *implies* $x \in \bigcup \mathfrak{G}$.

Accordingly, a class $\mathfrak{F} \subset C(H_\alpha, \mathcal{N})$ will be called *perfect* iff it satisfies the above condition. We have

4.3. *If $\mathfrak{F} \subset C(H_\alpha, \mathcal{N})$ is perfect, then every $\mathfrak{F}^*$ with $\mathfrak{F} \subset \mathfrak{F}^* \subset C(H_\alpha, \mathcal{N})$ is perfect.*

The above is, of course, a restatement of the known fact that for any space X and for $\mathfrak{F} \subset \mathfrak{F}^* \subset C(X, \mathcal{N})$, $\mathbb{P}\mathfrak{F}$ is perfect implies that $\mathbb{P}\mathfrak{F}^*$ is perfect. Topological proof of this fact can be (routinely) translated into set-theoretic terms; since this translation is short and not devoid of a certain infantile charm, we will give it in detail.

Set-theoretic proof of 4.3. Treatment of condition a) is obvious.

Assume that $\mathfrak{G}$ satisfies the assumptions of b) relative to $\mathfrak{F}^*$. If $\mathfrak{G}$ satisfies the assumptions of b) relative to $\mathfrak{F}$, then there is nothing to prove. Assume therefore that there are x's so that $f(x) = \phi(f)$ for every $f \in \mathfrak{F}$ but $x \notin \bigcup \mathfrak{G}$; by a), there is only finitely many of them, we will list them as $x_1, \dots, x_k$. For each i select a function $f_i \in \mathfrak{F}^*$ so that $f_i(x_i) \neq \phi(f_i)$ and, in turn, select an open interval I_i so that $x_i \in I_i$ and f_i is constant on I_i. The collection $\mathfrak{G} \cup \{f_1, \dots, f_k\}$ satisfies the assumptions of b) relative to $\mathfrak{F}$; let $f'_1, \dots, f'_l$ be the functions satisfying the conclusion of b) relative to $\mathfrak{F}$. Taking the system $\{f_i\}_i \cup \{f'_j\}_j$ we end the proof. Indeed, $f'_j(x) = \phi(f_j)$ for $j = 1, \dots, l$ implies $x \in \bigcup \mathfrak{G} \cup I_1 \cup \dots \cup I_k$. But $f_i(x) = \phi(f_i)$ implies $f_i(x) \neq f_i(x_i)$ and since f_i is constant on I_i, we have $f_i(x) \neq f_i(x')$ for every $x' \in I_i$; thus $x \notin I_i$. Therefore $x \in \bigcup \mathfrak{G}$.

Now, the set-theoretic translation of 1.1a (i.e., of 1.1 with "hereditary" removed) is as follows:

4.4. *H_α admits a perfect class $\mathfrak{F} \subset C(H_\alpha, \mathcal{N})$ (equivalently, by 4.3, the class $C(H_\alpha, \mathcal{N})$ is perfect); further, if $\alpha = \beta + 1$ and $2^{\aleph_\beta} = \aleph_\alpha$, then every perfect class $\mathfrak{F} \subset C(H_\alpha, \mathcal{N})$ has cardinality $> \aleph_\alpha$.*

C. 2nd countable $\mathcal{N}$-compact spaces. 2nd countable $\mathcal{N}$-compact spaces are simply 0-dimensional separable metric space, i.e., subspaces of the Cantor set $\mathcal{C}$. They are strongly 0-dimensional – i.e., the covering dimension, dim, is equal to 0; therefore their $\mathcal{N}$- and $\mathcal{R}$-defects are the same. Despite the simplicity of these spaces the situation concerning their $\mathcal{N}$-defect is of some degree of intricacy. The matters started with the question stated (somewhat casually) in the last sentence of [M2]: (given that $\operatorname{def}_{\mathcal{N}} Q > \aleph_0$) ... *I do not know if one can prove, without the continuum hypothesis, that* $\operatorname{def}_{\mathcal{N}} Q = 2^{\aleph_0}$? This (and more) have been answered in the negative by Heckler [Hec] (using previous result of Katetov). Heckler results, together with their expansion given by van Douwen [vDou], cover the situation of 2nd countable spaces quite completely and below we present their summary (with proofs).

If we let $\mathfrak{k}(X)$ = the smallest cardinal $\mathfrak{k}$ so that X is the union of $\leq \mathfrak{k}$ compact subsets, then obviously $\mathfrak{k}(cX \setminus X) = \mathfrak{k}(\beta X \setminus X)$ for every compactification cX of X. This, combined with 4.2, shows (as pointed out by van Douwen) that if cX is 0-dimensional and perfectly normal, then $\operatorname{def}_N X = \mathfrak{k}(cX \setminus X)$. In particular, if X is a dense subset of the Cantor set $\mathcal{C}$, then $\operatorname{def}_N X = \mathfrak{k}(\mathcal{C} \setminus X)$. It follows (Heckler) that $\operatorname{def}_{\mathcal{N}} \mathcal{Q} = \mathfrak{k}(\mathcal{P})$ (view here $\mathcal{Q}$ as the set of endpoints of $\mathcal{C}$ and $\mathcal{P}$ as $\mathcal{C} \setminus \mathcal{Q}$). Viewing $\mathcal{P}$ as the set of $\mathcal{N}$-valued sequences $\mathcal{N}^{\aleph_0}$ and letting, for a $t \in \mathcal{P}$, $\mathcal{P}(t) = \{s \in \mathcal{P} : s \leq t\}$, we see that the class $\{\mathcal{P}(t) : t \in \mathcal{P}\}$ is cofinal (inclusionwise) with

the class of all compact subsets of $\mathcal{P}$. Consequently (Katetov), $\mathfrak{k}(\mathcal{P})$ is the smallest of the cardinalities of sets $A \subset \mathcal{P}$ with $\mathcal{P} = \bigcup\{\mathcal{P}(t) : t \in A\}$; i.e., $\mathfrak{k}(\mathcal{P})$ is what is sometimes called the Rothberger aleph $\aleph_R$. $\aleph_R$ cannot be determined in ZFC, hence (Heckler) $\mathrm{def}_{\mathcal{N}}\mathcal{Q}$ *cannot be determined in ZFC*; more precisely, "$\aleph_1 < 2^{\aleph_0}$" *is consistent with each of these three statements* "$\mathrm{def}_{\mathcal{N}}\mathcal{Q} = \aleph_1$", "$\aleph_1 < \mathrm{def}_{\mathcal{N}}\mathcal{Q} < 2^{\aleph_0}$", "$\mathrm{def}_{\mathcal{N}}\mathcal{Q} = 2^{\aleph_0}$". Since $\mathfrak{k}(A) \le \aleph_R$ holds for every $A \subset \mathcal{C}$ which is a continuous image of $\mathcal{P}$ (i.e., for every analytic set $A \subset \mathcal{C}$), we have (as observed by van Douwen, answering a question of Heckler) that *for any dense subset X of $\mathcal{C}$ with analytic $\mathcal{C} \setminus X$,* $\mathrm{def}_N X$ *can be determined in ZFC if and only if* $\mathrm{def}_N X \le \aleph_0$*; i.e., iff X is a $\mathbf{G}_\delta$-set.*

On the other hand, for every $\mathfrak{m}$ with $\aleph_0 \le \mathfrak{m} \le 2^{\aleph_0}$ there is a dense subset B of $\mathcal{C}$ with $\mathrm{card}\, B = \mathfrak{m}$ and having only countable compact subsets (trivial if $\mathfrak{m} < 2^{\aleph_0}$, for $\mathfrak{m} = 2^{\aleph_0}$ use Bernstein sets); thus (van Douwen) *for every such $\mathfrak{m}$ there is an X with* $\mathrm{def}_N X = \mathfrak{m}$. Hence we are witnessing here the ubiquitous dichotomy between the simple and the arcane[3]: $\mathcal{N}$-defect of an "effectively" given set X can be determined in ZFC only in the trivial case; on the other hand, sets with an arbitrary (within bounds $\aleph_0 < \mathfrak{m} \le 2^{\aleph_0}$) prescribed $\mathcal{N}$-defect can be "constructed" by ineffective means.

An obvious question is which of the above results remain true for $\mathrm{ext}_{\mathcal{N}} X$. In most cases this would be strengthening of these results. In particular, *is the existence of 2nd countable spaces with* $\aleph_0 < \mathrm{ext}_{\mathcal{N}} X < 2^{\aleph_0}$ *consistent with* ZFC? Better yet, *is* $\mathrm{ext}_{\mathcal{N}}\mathcal{Q} < 2^{\aleph_0}$ *consistent with* ZFC?

D. $\mathcal{N}$-compact spaces with weight $> \aleph_0$**.** Here our knowledge is very limited; accordingly, in contrast to the previous subsection, this subsection contains not results but problems. Prior to this paper the only known result on the $\mathcal{N}$-defect of spaces with weight $> \aleph_0$ was the statement that $\mathrm{def}_{\mathcal{N}}\mathcal{D}_{\mathfrak{m}} = \mathfrak{m}$ for most Ulam non-measurable $\mathfrak{m}$ [M1]; $\mathcal{D}_{\mathfrak{m}}$ is the discrete space of cardinality $\mathfrak{m}$. This result, as well as all the results mentioned in subsection C, were, in principle, reinterpretations of known set-theoretic facts; it appears that further progress in this direction will require, or will generate – perhaps implicitly – new set-theoretic facts (see the preceding subsection).

The basic problem, still unanswered even for the smallest uncountable cardinal $\mathfrak{m}$, is the existence, in ZFC, of $\mathcal{R}$-compact spaces X with the given $\exp_{\mathcal{R}} X = \mathfrak{m}$ and $\mathrm{Exp}_{\mathcal{R}} X > \exp_{\mathcal{R}} X$ (and the same with $\mathcal{R}$ replaced by $\mathcal{N}$). Beyond that, there are two broad classes of problems. First is to determine the $\mathcal{N}$- and $\mathcal{R}$-defect of known spaces. "Determine" is meant here in a very general sense; for instance, the fact that the defect of a given space cannot be determined in ZFC is certainly a result in this direction. Thus one would expect that $\mathrm{def}_{\mathcal{N}}\mathcal{Q}(\mathfrak{m})$ cannot be determined in ZFC. Better yet, although $\mathrm{def}_N Q > \aleph_0$ is a statement of ZFC, $\mathrm{def}_{\mathcal{N}}\mathcal{Q}(\mathfrak{m}) > \mathfrak{m}$ is not. Thus it is natural to ask *is it consistent with ZFC that any cardinal $\mathfrak{n}$ with $m \le n \le 2^{\mathfrak{m}}$ can be the value of* $\mathrm{def}_{\mathcal{N}}\mathcal{Q}(\mathfrak{m})$? and further *is it consistent with ZFC that* $\mathrm{def}_{\mathcal{N}}\mathcal{Q}(\mathfrak{m}) < \mathrm{weight}\,\mathcal{Q}(\mathfrak{m})$?

[3]Our discussion would not be complete without this quotation: "*In many cases of such dichotomizing, the message that gets across to the reader is chiefly that the writer is using a fancy, academic-sounding word. If this is the impression you want to convey, dichotomy will surely serve you. If you are mainly interested in having your sentence understood, however, you might be better off finding another way to word it.*" (Merriam Webster's Dictionary of English Usage).

Continuing, we have an obvious problem concerning the inequality $\operatorname{def}_{\mathcal{R}} X \leq \operatorname{def}_{\mathcal{N}} X$: *can this inequality be strict*? This can happen only for $\mathcal{N}$-compact spaces with $\dim > 0$. Thus it is particularly interesting to examine the $\mathcal{R}$- and $\mathcal{N}$-defect of such spaces. At the present time examples of such spaces are quite numerous and some of them are rather simple. But, as we have seen, the simplicity of a space does not guarantee that the determination of the defect is simple. It would be still more interesting to realize the inequality $\operatorname{def}_{\mathcal{R}} X < \operatorname{def}_{\mathcal{N}} X$ within a given subclass of $\mathcal{N}$-compact spaces. In this respect the following classes are of particular interest: the class of CO-Borel spaces (i.e., the spaces in which every functionally closed set belongs to the σ-algebra generated by clopen sets)[4] and the class of metric spaces. We note that the involvement of metric spaces in these questions probably makes the matters extremely difficult. The supply of metric $\mathcal{N}$-compact spaces with $\dim > 0$ is still very limited (the first was given in [M4]; the only other source is [Kul]) and they cannot be called simple. In particular, it is not known if any of them is CO-Borel. (Kulesza told me that one of his $\mathcal{N}$-compact metric spaces in [Kul] is not CO-Borel. He has also shown that the space $\nu\mu_0$ of [M6] is $\mathcal{N}$-compact. Since $\nu\mu_0$ is, at the present moment, the most pathological example, one could speculate that it is most likely to satisfy, within metric spaces, the inequality $\operatorname{def}_{\mathcal{R}} X < \operatorname{def}_{\mathcal{N}} X$.) Thus if the class of CO-Borel $\mathcal{N}$-compact metric spaces is non-empty (which appears to be very likely and the answer will probably be produced quite soon), then the realization of the inequality $\operatorname{def}_{\mathcal{R}} X < \operatorname{def}_{\mathcal{N}} X$ within this class would be the best result in this direction.

The second group of problems is to find spaces with prescribed $\mathcal{R}$- and $\mathcal{N}$-defects – this could require construction of new spaces. A typical problem is to investigate the extent of the set $\{\operatorname{def}_{\mathcal{N}} X : \operatorname{weight} X = \mathfrak{m}\}$. *Can we prove (in ZFC or under additional conditions) that this set contains all cardinals* $\mathfrak{m}$ *with* $\aleph_0 < \mathfrak{m} \leq 2^{\mathfrak{m}}$? (The case of $\operatorname{def}_{\mathcal{N}} X \leq \aleph_0$ is trivial: if $\operatorname{def}_{\mathcal{N}} X \leq \aleph_0$, then $\operatorname{def}_{\mathcal{N}} X$ is equal to one of the numbers 0, 1, $\aleph_0$; 0 – iff X is compact, 1 – iff X is not compact, but it is (homeomorphic to) a functionally open subset of $\mathcal{D}^{\mathfrak{m}}$, $\aleph_0$ – iff X is none of the above, but it is a countable intersection of functionally open subsets of $\mathcal{D}^{\mathfrak{m}}$. Thus $\{\operatorname{def}_{\mathcal{N}} X : \operatorname{weight} X = \mathfrak{m}\}$ always contains the cardinals 0, 1, $\aleph_0$.)

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